Robust stability of fractional-order nonlinear systems under sliding mode controller with fractional-order reaching law

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Abstract: In this paper, the problem of robust stability of fractional-order nonlinear systems under sliding mode control (SMC) with fractional-order (FO) switching law is discussed. The proposed FO switching law, involving an FO derivative function, is proven to guarantee that the reaching phase can happen in finite time. The calculation formula of the reaching time is computed. The comparisons between FO and integer-order (IO) switching laws reveal the potential advantages of one controller over the other. The stability criterion of the sliding mode dynamics is derived in terms of linear matrix inequalities (LMIs). The tradeoff between control performance and parameters selection is discussed and visualized. Simulation results are presented to illustrate the effectiveness of the designed FO SMC.

Key Words: Sliding mode technique; Fractional order reaching law; Reaching time; Fractional-order integral switching surface; Robust stability

1 INTRODUCTION

Fractional calculus deals with FO differentiation and integration [1, 2]. Most of the real systems are actually with FO dynamics, such as viscoelastic fluids, chaos and fractals. FO controllers, as a generalization of IO controllers, have a greater flexility in improving the performance of the controlled system. This possible benefits have attracted interest in a great variety of FO control [3, 4, 5, 6, 7, 8].

SMC [9, 10, 11, 12, 13] is known as a robust method to control nonlinear systems operating under uncertainty conditions. The main feature of SMC is to switch the control law to force the states of the system from the initial states onto some predefined sliding surface. Most published results about FO SMC are limited to FO linear systems via SMC [13], FO chaotic systems under SMC [14, 15, 16, 17, 18] and some application of FO SMC [8, 19, 20]. However, one main problem is how to develop direct systematic methods for designing SMC for FO nonlinear systems. Although this scheme has been discussed in the recent work [21], it needs to further explore the properties of FO SMC for FO nonlinear systems. It should be analyzed why and how to obtain a better control performance for FO SMC.

With this motivation, a novel FO SMC including FO switching law and fractional integral sliding surface is proposed for uncertain FO nonlinear systems. Some effective tools for FO control analysis are given. A concept of the FO sign function $D_t^q \mathrm{sgn}(s), 0 \leq q < 1$, involving an FO differentiator, is introduced to building a switching law. Similar to the sign function, $D_t^q \mathrm{sgn}(s)$ is proven to be able to extract the sign of s in this paper. It can ensure the occurrence of the reaching phase in finite time. More importantly, *the calculation formula of the reaching time* t_{reach} under FO switching law is derived, for the first time. The comparison between FO and IO switching laws reveals the potential advantages of one controller over the other. For the sliding phase, a fractional integral sliding surface is developed for FO nonlinear systems, based on our previous work [22]. The sliding mode dynamics is analyzed to achieve to stability condition. Finally, simulation results demonstrate the advantages of the designed control scheme.

Notations: $A < 0$ means that A is a real symmetric negative definitive matrix. A^T and A^{-1} represent the transpose and the inverse of matrix A. I is the identity matrix with appropriate dimensions. $\mathrm{diag}\{\cdots\}$ denotes the block diagonal matrix. $\otimes$ is the Kronecker product of two matrices and $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$. $Sym\{X\}$ denotes the expression $X^T + X$.

2 FORMATTING INSTRUCTIONS

Consider the following uncertain FO nonlinear system

$$\begin{aligned} D_t^\beta x(t) &= (A + \delta(t))x(t) + Ff(x,t) + Bu(t) + B_d\omega(t), \\ y(t) &= \Psi x(t), \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^\mu, y(t) \in \mathbb{R}^\upsilon$ are the state vector, the control input and the output. $f(x,t) \in \mathbb{R}^l$ is the nonlinear function. $\omega(t)$ denotes the external disturbance, which is assumed to be bounded (i.e. there exists a positive constant d such that $\|\omega(t)\| < d$). $A \in \mathbb{R}^{n\times n}, F \in \mathbb{R}^{n\times l}, B \in \mathbb{R}^{n\times\mu}, B_d \in \mathbb{R}^{n\times\mu}, \Psi \in \mathbb{R}^{\upsilon\times n}$ are constant

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known matrices. β is the fractional commensurate order satisfying $0<\beta<1$. The time-varying uncertainty $\delta(t)$ is assumed to be $\delta(t)=WU(t)N$, where W, N are known real constant matrices, the time-varying continuous function $U(t)$ satisfies $\|U(t)\|<1, \forall t>0$. Before deriving the main results, we need the following definition and Lemmas:

Definition 2.1 *The Riemann-Liouville fractional integral with α is defined as*

$$D_t^{-\alpha}\zeta(t)=I_t^{\alpha}\zeta(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\zeta(\tau)d\tau, \quad (2)$$

where $\zeta(t)$ is an arbitrary function, I_t^{α} is the fractional integral of order α on $[0,t]$, and Γ is the Gamma function. Likewise, the Riemann-Liouville definition of the βth-order fractional derivative operator is defined as

$$D_t^{\beta}\zeta(t)=\frac{1}{\Gamma(m-\beta)}(\frac{d}{dt})^m\int_0^t\frac{\zeta(\tau)}{(t-\tau)^{1+\beta-m}}d\tau, \quad (3)$$

where m is the first integer larger than β, i.e. $m-1\leq\beta<m$. The Caputo definition is

$$D_t^{\beta}\zeta(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t\frac{\zeta^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}}d\tau. \quad (4)$$

Lemma 2.1 [8] *For $D_t^q\sigma(t)=\frac{1}{\Gamma(1-q)}\frac{d}{dt}\int_0^t\frac{\sigma(\tau)}{(t-\tau)^q}d\tau, 0\leq q<1$ and sign function, one can conclude that*

$$D_t^q\mathrm{sgn}(\sigma(t))\begin{cases}>0, & \text{if} \quad \sigma(t)>0, \quad \text{when} \quad t>0,\\ <0, & \text{if} \quad \sigma(t)<0, \quad \text{when} \quad t>0,\end{cases} \quad (5)$$

Remark 2.1 A concept of the FO sign function $D_t^q\mathrm{sgn}(\sigma), 0\leq q<1$, including an FO differentiator, is proposed. Alike to the sign function, $D_t^q\mathrm{sgn}(\sigma)$ is demonstrated to be capable of extracting the sign of σ. One may think it is trivial compared with the sign function itself; others may doubt that it is against instinct compared with the derivative of a generic function. The meaning of $D_t^q\mathrm{sgn}(\sigma), 0\leq q<1$ is the **FO derivative** of the sign function. The sign of the derivative is generally not the same as the sign of the function itself. However, the sign of $D_t^q\mathrm{sgn}(\sigma), 0\leq q<1$ is shown to be the same as $\mathrm{sgn}(\sigma)$ in this paper. This is the important property of the FO sign function.

Remark 2.2 The FO sign function will be applied to build an FO switching function. According to Lemma 2.1, the corresponding FO switching function can guarantee to force the system on the sliding surface. It will be shown in the following.

Lemma 2.2 For any matrices X and Y with appropriate dimensions, one has

$$X^TY+Y^TX\leq(1/\lambda)X^TX+\lambda Y^TY, \quad \forall\lambda>0. \quad (6)$$

Lemma 2.4 Let $\Xi\in\mathbb{R}^{n\times n}, 0<\beta<1$ and $\theta=(1-\beta)(\pi/2)$. The FO system $(d^{\beta}x(t)/dt^{\beta})=\Xi x(t)$ is asymptotically stable if and only if there exist two positive definite Hermitian matrices $Q_1=Q_1^*\in\mathbb{C}^{n\times n}$ and $Q_2=Q_2^*\in\mathbb{C}^{n\times n}$ such that $e^{\theta i}Q_1\Xi^*+e^{-\theta i}\Xi Q_1+e^{-\theta i}Q_2\Xi^*+e^{\theta i}\Xi Q_2<0$ or $e^{\theta i}Q_1\Xi+e^{-\theta i}\Xi^*Q_1+e^{-\theta i}Q_2\Xi+e^{\theta i}\Xi^*Q_2<0$.

Fact 2.1 A complex Hermitian matrix Q satisfies $Q<0$ if and only if

$$\begin{bmatrix}\mathrm{Re}(Q) & \mathrm{Im}(Q)\\ -\mathrm{Im}(Q) & \mathrm{Re}(Q)\end{bmatrix}<0. \quad (7)$$

3 Sliding surface and control scheme design

In this paper, an FO integral sliding surface is built

$$s=D_t^{\beta-1}(C_1x+C_2z), \quad (8)$$

in which $D_t^{\beta}z=Kx-z$ with $s\in\mathbb{R}^{\mu}, z\in\mathbb{R}^{\iota}, C_1\in\mathbb{R}^{\mu\times n}, C_2\in\mathbb{R}^{\mu\times\iota}, K\in\mathbb{R}^{\iota\times n}$. Based on the IO and FO sign function, $\mathrm{sgn}(s):\mathbb{R}^{\mu}\to\mathbb{R}^{\mu}, D_t^q\mathrm{sgn}(s):\mathbb{R}^{\mu}\to\mathbb{R}^{\mu}$ are defined by

$$\begin{aligned}\mathrm{sgn}(s)&=[\mathrm{sgn}(s_1),\mathrm{sgn}(s_2),\cdots,\mathrm{sgn}(s_{\mu})]^T,\\ D_t^q\mathrm{sgn}(s)&=[D_t^q\mathrm{sgn}(s_1),\cdots,D_t^q\mathrm{sgn}(s_{\mu})]^T, \end{aligned} \quad (9)$$

where q satisfies $0\leq q<1$. Then, from (9), a new FO switching law is proposed as follows:

$$u_{reach}=-HD_t^q\mathrm{sgn}(s), \quad (10)$$

where $H=\mathrm{diag}(h_1,h_2,\cdots,h_{\mu})$ with $h_i,(i=1,2,\cdots,\mu)$ are positive constants.

Remark 3.1 The parameter q is the fractional-order of the FO sign function $D_t^q\mathrm{sgn}(s), 0\leq q<1$. The parameter q can be tuned to obtain a shorter reaching time and better control performance. Note that $\mathrm{sgn}(s)$ is a special case of $D_t^q\mathrm{sgn}(s)$. Therefore, the classical IO switching law should be equivalent to the above FO switching law (10) with $q=0$. Hence, there is a better flexibility in adjusting FO than IO switching law.

In this case, the sliding mode control law is given

$$u=-(C_1B)^{-1}[(C_1A+C_2K+C_1)x+C_1Ff(x,t)+\bar{u}], \quad (11)$$

in which $\bar{u}=w_1-D_t^{1-\beta}s$ with $w_1=\|C_1W\|\,\|Nx\|\,\mathrm{sgn}(s)+d\|C_1B_d\|\mathrm{sgn}(s)+HD_t^q\mathrm{sgn}(s)$.

In the following, the reachability condition of the sliding surface will be considered.

Theorem 3.1 Consider the FO nonlinear system (7) and sliding surface function (8), the trajectories of the system (7) under the controller (11) with the FO switching law (10) can be driven onto the sliding surface $s(t)=0$.

Proof. Consider the following $V(t)=s^T(t)s(t)$. Taking the fractional differentiating with respect to time, one has

$$\dot{V}=[C_1D_t^{\beta}x+C_2D_t^{\beta}z]^Ts+s^T[C_1D_t^{\beta}x+C_2D_t^{\beta}z]. \quad (12)$$

Substitution of (7) and (11) into (12) yields

$$\begin{aligned}\dot{V}&=[C_1\delta(t)x+C_1B_d\omega(t)-w_1]^Ts\\ &\quad+s^T[C_1\delta(t)x+C_1B_d\omega(t)-w_1]. \end{aligned} \quad (13)$$

Since $\|G(t)\| < 1$, one has

$$\begin{aligned}\dot{V} &= \Omega + \Omega^T - (s^T H D_t^q \mathrm{sgn}(s) + (H D_t^q \mathrm{sgn}(s))^T s) \\ &\leq -(s^T H D_t^q \mathrm{sgn}(s) + (H D_t^q \mathrm{sgn}(s))^T s),\end{aligned} \tag{14}$$

where $\Omega = s^T C_1 \delta(t) x + s^T C_1 B_d \omega(t) - s^T \|C_1 W\| \|Nx\| \mathrm{sgn}(s) - s^T d \|C_1 B_d\| \mathrm{sgn}(s)$.

According to Lemma 2.1, one has $\dot{V} < 0$. One can conclude that the state trajectories of the system (7) under the controller (11) hit the sliding surface. Hence, the trajectories of the system can be driven onto the predefined sliding surface. The proof is completed.

Finite reaching time is one of the main characteristics of SMC. The calculation formula of reaching time will be explained later. First, we consider the sliding surface $s(t) \in \mathbb{R}$.

Hence, we have $0.5\dot{V} = s\dot{s} \leq -hsD_t^q \mathrm{sgn}(s)$, in which h is a positive constant. Let $s\dot{s} = -hsD_t^q \mathrm{sgn}(s)$, consider two cases:

1) when the initial condition $s(0)$ is greater than zero ($s(0) > 0$),

$$\dot{s} = -hD_t^q \mathrm{sgn}(s). \tag{15}$$

Hence, $s(t) = -\frac{ht^{1-q}}{(1-q)\Gamma(1-q)} + s(0)$. At $t = t_{reach}$, $s(t_{reach}) = 0$. Thus, one has

$$t_{reach} = \left(\frac{s(0)(1-q)\Gamma(1-q)}{h} \right)^{1/(1-q)}. \tag{16}$$

2) When the initial condition $s(t)$ is less than zero ($s(0) < 0$),

$$\dot{s} = -hD_t^q \mathrm{sgn}(s). \tag{17}$$

Hence, $s(t) = \frac{ht^{1-q}}{(1-q)\Gamma(1-q)} + s(0)$. At $t = t_{reach}$, $s(t_{reach}) = 0$. Thus, one has

$$t_{reach} = \left(\frac{-s(0)(1-q)\Gamma(1-q)}{h} \right)^{1/(1-q)}. \tag{18}$$

According to **1)**-**2)**, one can obtain

$$t_{reach} = \left(\frac{|s(0)|(1-q)\Gamma(1-q)}{h} \right)^{1/(1-q)}. \tag{19}$$

Remark 3.2 *Considering the IO switching law $u_{reach} = -h\mathrm{sgn}(s)$, there are two cases: 3) when $s(0) > 0$, $t_{reach} = s(0)/h$; 4) when $s(0) < 0$, $t_{reach} = -s(0)/h$. Thus, the reaching time under the IO switching law is $t_{reach} = |s(0)|/h$.*

Remark 3.3 *Due to $D_t^0 \mathrm{sgn}(s) = \mathrm{sgn}(s)$, it is more likely for the FO switching law to obtain better performance. According to (19), the parameters $s(0), h$ and q have influence on t_{reach}. Figure 1 shows the changes in t_{reach} along with $h/|s(0)|$ and q, in which $h/|s(0)| \in [1, 5]$ and $q \in [0, 0.98]$. One has that t_{reach} via the FO switching law with $0 < q \leq 0.98$ is smaller than that the IO switching law does, when $1 \leq h/|s(0)| \leq 5$. For the other conditions, the smaller can be derived by analyzing relationship among $t_{reach}, s(0), h, q$, similar to the above case.*

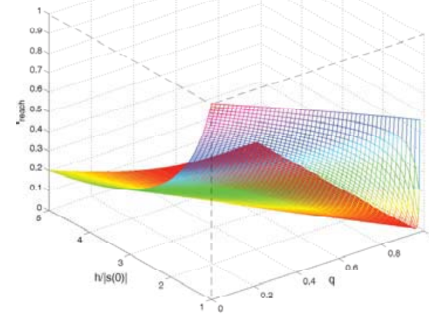

Figure 1: The relationship between t_{reach}, q and $h/|s(0)|$, when considering the FO switching law.

For the switching surface $s = [s_1(t), s_2(t), \cdots, s_\mu(t)]^T \in \mathbb{R}^\mu$, the reaching time t^i_{reach} for $s_i(t), (i = 1, 2, \cdots, \mu)$ can be derived as the same steps as the above analysis

$$t^i_{reach} = \left(\frac{|s_i(0)|(1-q)\Gamma(1-q)}{h_i} \right)^{1/(1-q)}. \tag{20}$$

Hence, the reach time is $t_{reach} = \max\{t^i_{reach}, i = 1, 2, \cdots, \mu\}$.

Next, the stability analysis of the sliding mode dynamics will be investigated. Substituting the controller (11) into the system (7), the sliding mode dynamics can be obtained

$$\begin{aligned}&D_t^\beta x(t) = (-C_1^{-1} C_2 K - I + \delta(t)) x(t), \\ &y(t) = \Psi x(t).\end{aligned} \tag{21}$$

Theorem 4.2 *The FO sliding mode dynamics (21) is asymptotically stabilization if there exist two real symmetric matrices $G_{k1} \in \mathbb{R}^{n \times n}, k = 1, 2$, and two skew-symmetric matrices $G_{k2} \in \mathbb{R}^{n \times n}, k = 1, 2$, and real scalar scalars $\lambda_{ij} > 0, (i, j = 1, 2)$, such that*

$$\begin{bmatrix} \Xi_{11} & * & * & * & \\ I_2 \otimes (G_{11} W) & \Xi_{22} & * & * & * \\ I_2 \otimes (G_{12} W) & 0 & \Xi_{33} & * & * \\ I_2 \otimes (G_{21} W) & 0 & 0 & \Xi_{44} & * \\ I_2 \otimes (G_{22} W) & 0 & 0 & 0 & \Xi_{55} \end{bmatrix} < 0, \tag{22}$$

$$\begin{bmatrix} G_{11} & G_{12} \\ -G_{12} & G_{11} \end{bmatrix} > 0, \begin{bmatrix} G_{21} & G_{22} \\ -G_{22} & G_{21} \end{bmatrix} > 0, \tag{23}$$

where

$$
\begin{aligned}
&\Xi_{11}=\sum_{i=1}^{2}\sum_{j=1}^{2}\{Sym\{\Lambda_{ij}\otimes G_{ij}(-C_1^{-1}C_2K-I)\}\\
&+\lambda_{ij}(I_2\otimes N^TN)\},\Xi_{22}=-\lambda_{11}I_{2n},\Xi_{33}=-\lambda_{12}I_{2n},\\
&\Xi_{44}=-\lambda_{21}I_{2n},\Xi_{55}=-\lambda_{22}I_{2n},\\
&\Lambda_{11}=\begin{bmatrix}\sin(\frac{\pi}{2}\beta) & \cos(\frac{\pi}{2}\beta)\\ -\cos(\frac{\pi}{2}\beta) & \sin(\frac{\pi}{2}\beta)\end{bmatrix},\\
&\Lambda_{12}=\begin{bmatrix}-\cos(\frac{\pi}{2}\beta) & \sin(\frac{\pi}{2}\beta)\\ -\sin(\frac{\pi}{2}\beta) & -\cos(\frac{\pi}{2}\beta)\end{bmatrix},\\
&\Lambda_{21}=\begin{bmatrix}\sin(\frac{\pi}{2}\beta) & -\cos(\frac{\pi}{2}\beta)\\ \cos(\frac{\pi}{2}\beta) & \sin(\frac{\pi}{2}\beta)\end{bmatrix},\\
&\Lambda_{22}=\begin{bmatrix}\cos(\frac{\pi}{2}\beta) & \sin(\frac{\pi}{2}\beta)\\ -\sin(\frac{\pi}{2}\beta) & \cos(\frac{\pi}{2}\beta)\end{bmatrix}.
\end{aligned}
$$

Proof. Since $\delta(t)=WU(t)N$, the sliding mode dynamics can be rewritten as

$$
\begin{aligned}
&D_t^{\beta}x(t)=(-C_1^{-1}C_2K-I+WU(t)N)x(t),\\
&y(t)=\Psi x(t).
\end{aligned}\tag{24}
$$

Suppose that there exist two real symmetric matrices $G_{k1}\in\mathbb{R}^{n\times n},k=1,2$, and two skew-symmetric matrices $G_{k2}\in\mathbb{R}^{n\times n},k=1,2$, and real scalar scalars $\lambda_{ij}>0,(i,j=1,2)$, such that (22) and (23) hold. It follows from Lemma 2.4 that

$$
\begin{aligned}
&\sum_{i=1}^{2}\sum_{j=1}^{2}Sym\{\Lambda_{ij}\otimes G_{ij}(-C_1^{-1}C_2K-I+\delta(t))\}\\
=&\sum_{i=1}^{2}\sum_{j=1}^{2}Sym\{\Lambda_{ij}\otimes(G_{ij}(-C_1^{-1}C_2K-I)\}\\
&+\sum_{i=1}^{2}\sum_{j=1}^{2}Sym\{\Lambda_{ij}\otimes(G_{ij}WU(t)N)\}.
\end{aligned}\tag{25}
$$

Note that the time-varying continuous function $U(t)$ implies

$$
\begin{aligned}
(I_2\otimes U(t))(I_2\otimes U(t))^T&=(I_2\otimes U(t))(I_2\otimes U^T(t))\\
&=I_2\otimes(U(t)U^T(t))\le I.
\end{aligned}\tag{26}
$$

Since $\Lambda_{ij}^T\Lambda_{ij}=I_2,(i,j=1,2)$, it follows from (17) and Lemma 2.2 that for any real positive scalars $\lambda_{ij},(i,j=1,2)$

$$
\begin{aligned}
&Sym\{\Lambda_{ij}\otimes(G_{ij}WU(t)N)\}\\
=&Sym\{(I_2\otimes G_{ij}W)(I_2\otimes U(t))(\Lambda_{ij}\otimes N)\}\\
\le&\lambda_{ij}^{-1}(I_2\otimes G_{ij}W)(I_2\otimes G_{ij}W)^T+\lambda_{ij}(I_2\otimes N^TN).
\end{aligned}\tag{27}
$$

Substituting (27) into (25), one has

$$
\begin{aligned}
&\sum_{i=1}^{2}\sum_{j=1}^{2}Sym\{\Lambda_{ij}\otimes G_{ij}(-C_1^{-1}C_2K-I+\delta(t))\}\\
\le&\sum_{i=1}^{2}\sum_{j=1}^{2}Sym\{\Lambda_{ij}\otimes G_{ij}(-C_1^{-1}C_2K-I)\}\\
&+\sum_{i=1}^{2}\sum_{j=1}^{2}\{\lambda_{ij}(I_2\otimes N^TN)\}\\
&+\sum_{i=1}^{2}\sum_{j=1}^{2}\{\lambda_{ij}^{-1}(I_2\otimes G_{ij}W)(I_2\otimes G_{ij}W)^T\}.
\end{aligned}\tag{28}
$$

Taking (28) into account and using Lemma 2.2 for (22), the following inequality is obtained

$$
\sum_{i=1}^{2}\sum_{j=1}^{2}Sym\{\Lambda_{ij}\otimes G_{ij}(-C_1^{-1}C_2K-I+\delta(t))\}<0.\tag{29}
$$

From Fact 2.1, defining $\theta=(1-\beta)(\pi/2)$, one can conclude that

$$
\begin{aligned}
&Sym\{G_{11}(-C_1^{-1}C_2K-I+\delta(t))\cos\theta\\
&-G_{12}(-C_1^{-1}C_2K-I+\delta(t))\sin\theta\\
&+G_{21}(-C_1^{-1}C_2K-I+\delta(t))\cos\theta\\
&+G_{22}(-C_1^{-1}C_2K-I+\delta(t))\sin\theta\}\\
&+\mathrm{i}(G_{11}(-C_1^{-1}C_2K-I+\delta(t))\\
&-(-C_1^{-1}C_2K-I+\delta(t))^TG_{11})\sin\theta\\
&+\mathrm{i}(G_{12}(-C_1^{-1}C_2K-I+\delta(t))\\
&-(-C_1^{-1}C_2K-I+\delta(t))^TG_{12}^{\mathrm{T}})\cos\theta\\
&+\mathrm{i}((-C_1^{-1}C_2K-I+\delta(t))^TG_{21}\\
&-G_{21}(-C_1^{-1}C_2K-I+\delta(t)))\sin\theta\\
&+\mathrm{i}(G_{22}(-C_1^{-1}C_2K-I+\delta(t))\\
&-(-C_1^{-1}C_2K-I+\delta(t))^TG_{22}^{\mathrm{T}})\cos\theta<0.
\end{aligned}\tag{30}
$$

Furthermore, one has

$$
\begin{aligned}
&(\cos\theta+\mathrm{i}\sin\theta)(G_{11}+G_{12}\mathrm{i})(-C_1^{-1}C_2K-I+\delta(t))\\
&+(\cos\theta-\mathrm{i}\sin\theta)(-C_1^{-1}C_2K-I+\delta(t))^T(G_{11}-G_{12}^{\mathrm{T}}\mathrm{i})\\
&+(\cos\theta-\mathrm{i}\sin\theta)(G_{21}+G_{22}\mathrm{i})(-C_1^{-1}C_2K-I+\delta(t))\\
&+(\cos\theta+\mathrm{i}\sin\theta)(-C_1^{-1}C_2K-I+\delta(t))^T(G_{21}-G_{22}^{\mathrm{T}}\mathrm{i})\\
&<0,
\end{aligned}\tag{31}
$$

On the other hand, from Fact 2.1, (31) is equivalent to

$$
G_{11}+G_{12}\mathrm{i}>0,G_{21}+G_{22}\mathrm{i}>0.\tag{32}
$$

Let $G_{k1}=\mathrm{Re}(Q_k),G_{k2}=\mathrm{Im}(Q_k),\ k=1,2$. Hence, one has that $Q_1=Q_1^*\in\mathbb{C}^{n\times n}$ and $Q_2=Q_2^*\in\mathbb{C}^{n\times n}$ are Hermitian matrices. From (31) and (32), the following inequalities can be obtained

$$
\begin{aligned}
&e^{\theta\mathrm{i}}Q_1(-C_1^{-1}C_2K-I+\delta(t))\\
&+e^{-\theta\mathrm{i}}(-C_1^{-1}C_2K-I+\delta(t))^*Q_1\\
&+e^{-\theta\mathrm{i}}Q_2(-C_1^{-1}C_2K-I+\delta(t))\\
&+e^{\theta\mathrm{i}}(-C_1^{-1}C_2K-I+\delta(t))^*Q_2<0,
\end{aligned}\tag{33}
$$

$$
Q_1>0,Q_2>0.\tag{34}
$$

Hence, the sliding mode dynamics (21) is asymptotically stable. This completes our proof.

4 Numerical Simulation

In this section, one example is utilized to demonstrate the effectiveness and applicability of the proposed method.

Example 1. Consider the following system

$$\begin{aligned} D_t^{\beta} x = & \begin{bmatrix} a_{11} & -0.66 & 2.4 \\ -2.7 & a_{22} & 2.4 \\ -0.35 & 1.95 & a_{33} \end{bmatrix} x \\ & + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -0.3x_1x_3 \\ 0.1x_1x_2 \end{bmatrix} \\ & + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} u(t), \end{aligned} \quad (35)$$

in which $a_{11} = 2.75 + 0.1\sin(0.12t), a_{22} = 7.29 + 0.1\sin(0.12t), a_{33} = 9.78 + 0.1\sin(0.12t), \beta = 0.9$ and the uncertainty $\delta(t) = WU(t)N$ with

$$W = N = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$U(t) = \begin{bmatrix} 0.1\sin(0.12t) & 0 & 0 \\ 0 & 0.1\sin(0.12t) & 0 \\ 0 & 0 & 0.1\sin(0.12t) \end{bmatrix}.$$

Let

$$C_1 = \begin{bmatrix} -0.6483 & 0 & 0 \\ 0 & -0.6483 & 0 \\ 0 & 0 & -0.6483 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, H = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.2 \end{bmatrix}.$$

By using Theorem 3.2 to the system (35), a feasible solution of the symmetric matrices is found using MATLAB LMI Control Toolbox:

$$G_{11} = G_{21} = \begin{bmatrix} 1.3761 & 0.8911 & -1.3078 \\ 0.8911 & -2.5141 & -0.1986 \\ -1.3078 & -0.1986 & 1.9517 \end{bmatrix},$$

$$K = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, G_{12} = G_{22} = 0,$$

$$\lambda_{11} = 0.6343, \lambda_{12} = 0.5700, \lambda_{21} = 0.5700, \lambda_{22} = 0.5761,$$

By using (14), we can obtain the following SMC law:

$$\begin{aligned} u = & \begin{bmatrix} -5.2925 & 0.6600 & -2.4000 \\ 2.7000 & -9.8325 & -2.4000 \\ 0.3500 & -1.9500 & -12.3225 \end{bmatrix} x \\ & - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} f(x,t) \\ & + \begin{bmatrix} 1.5425 & 0 & 0 \\ 0 & 1.5425 & 0 \\ 0 & 0 & 1.5425 \end{bmatrix} \bar{u}, \end{aligned} \quad (36)$$

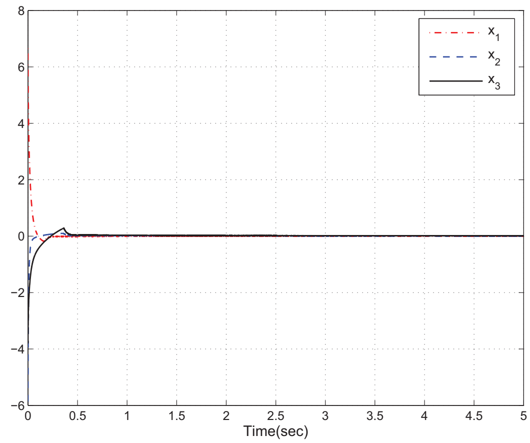

Figure 2: Time responses of states for the system (35) via the FO SMC (36).

in which $\bar{u} = 0.6211||Nx||\text{sgn}(s) + HD_t^q\text{sgn}(s) - D_t^{1-\beta}s$ with $q = 0.9$.

By Theorem 3.1, the system (35) under the controller (36), designed with the above parameters, converges to the sliding surface:

$$\begin{aligned} s = & D_t^{-0.1}\left\{ - \begin{bmatrix} 0.6483 & 0 & 0 \\ 0 & 0.6483 & 0 \\ 0 & 0 & 0.6483 \end{bmatrix} x \right. \\ & \left. + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} z \right\}, \\ D_t^{0.9} z = & \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} x - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} z. \end{aligned} \quad (37)$$

Figures 2-4 are the simulation results with the initial condition is $[x_1(0), x_2(0), x_3(0)]^T = [6.65, -5.86, -3.81]^T$. The fractional integration operator is approximated via Carlson method in frequency range (0.01,100) rad/s by using MATLAB toolbox called Ninteger. Figure 2 shows the time response of x_1, x_2, x_3 of the system under the controller (36) and the sliding surface (34). It shows that the controller (36) guarantees the states reaching to the sliding surface and final stabilization. Figure 3 shows the sliding surface (37). The control input (36) is depicted in Figure 4. It is obvious that the designed controller asymptotically stabilizes the unstable FO nonlinear system and the closed-loop system behavior is satisfactory.

5 Conclusions

The FO SMC with the FO switching law is proposed for a class of uncertain FO nonlinear systems. The proposed FO switching law is proven to ensure the occurrence of the reaching phase in finite time. The calculation formula of the reaching time is provided. The reachability analysis is discussed and visualized to show how to obtain a shorter reaching time. The stability analysis of the sliding mode dynamics is obtained by solving LMIs. Simulation results

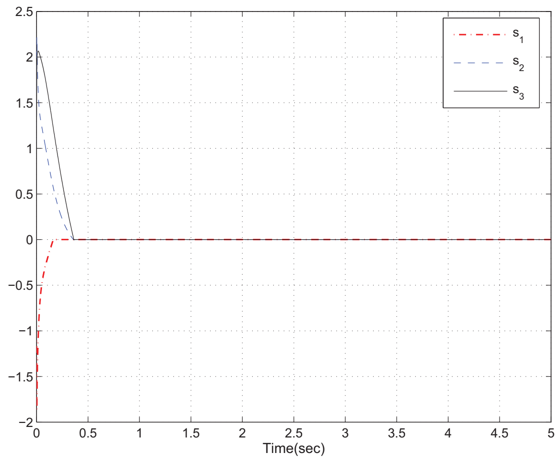

Figure 3: Time responses of the sliding surface (37).

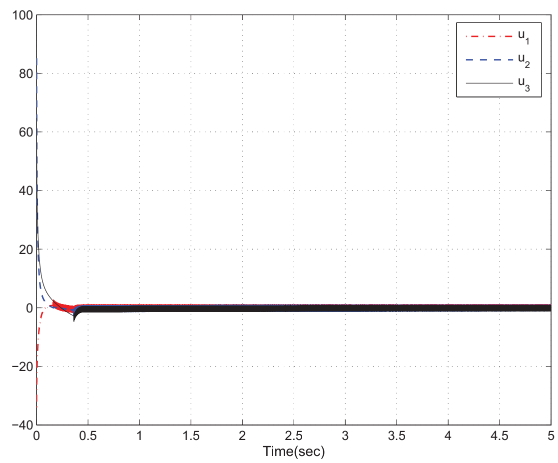

Figure 4: Time responses of the controller (36).

are presented to illustrate the effectiveness of the designed FO SMC.

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