Iterative Learning Control for Linear Discrete-Time Systems with Iteration-varying initial state and Packet Loss

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Abstract: This paper is concerned with the iterative learning control (ILC) design problem of linear discrete-time systems with iteration-varying initial state and packet loss. First, the iteration-varying initial state and the packet loss is introduced into the ILC process. In addition, the ILC design problem is transformed into a controller design problem of 2-D Roesser model. Next, the asymptotic stability of 2-D Roesser models under orthogonal initial states are investigated. Then, based on the 2-D asymptotic stability, a ILC algorithm is proposed to deal with the iteration-varying initial state and the packet loss. Finally, the effectiveness of the developed ILC algorithm is demonstrated by a simple example with various packet loss rate.

Key Words: Iterative learning control, linear discrete-time system, iteration-varying initial states, packet loss, 2-D system theory

1 INTRODUCTION

Iterative learning control (ILC) is an effective control scheme for dynamical system with repetitive operation over a fixed time interval. In general, an initial condition named as identical initialization condition ($i.i.c.$) ([19]) is indispensable to achieve the perfect tracking. However, for many practical control systems, $i.i.c.$ is hardly achievable. Therefore, the initial condition problem in ILC is frequently investigated ([11, 18]).

Recently, the 2-D system theory was successfully introduced to the ILC field ([4], [10]). However, the system invariance property is still a fundamental assumption for the perfect tracking. For instance, in [14], the robustness of ILC laws for a kind of iteration-varying initial states was discussed, but the bound of robustness could be large. It was shown in [6] that the perfect tracking can be fulfilled for iteration-varying initial states by using 2-D system theory, when the tracking time interval is sufficiently large.

In the last two decades, networked control systems have been widely applied due to their flexibility to deal with complex systems. However, because of the network congestion, the linkage interrupt, the transmission error, etc, the data packets could be lost in networked control systems which further may result in system performance degradation (see [12]). In addition, the packet loss is also considered in ILC field. For example, the iterative learning control is constructed for the discrete-time networked control systems with random packet losses and unknown control direction([17]).

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Our aim of this study is to analyze the ILC design problem with iteration-varying initial state and packet loss based on 2-D system theory. Compared to the existing literature, the main contributions of this work lie in the following:

(1) The 2-D system theory is used to deal with iteration-varying initial states. The proposed ILC algorithm can effectively deal with iteration-varying initial states such that the perfect tracking is achieved along the iterative axis.

(2) The ILC design problem is transformed into a controller design problem of s 2-D system. Based on the transformed 2-D ILC system, a novel approach is presented to handle the packet loss. If the data at t_0 and k_0 is lost, then the value of the state $\boldsymbol{x}_{k_0}(t_0)$, control $\boldsymbol{u}_{k_0}(t_0)$, output $\boldsymbol{y}_{k_0}(t_0+1)$ is set as that at t_0 and k_0-1, where t_0 and k_0 denote the given time and iteration. This approach ensures the the tracking performance of the ILC law is the same as that without packet loss, if the packet loss points are deleted in the convegence figures.

The note is organized as follows. Section 2 states the 2-D ILC model, introduces the packet loss problem. In Section 3, we establish the stability criteria of 2-D Roesser model under orthogonal initial states. In section 4, some sufficient conditions of convergence of the ILC rule are derived for initial conditions. In section 5, two numerical examples are presented to illustrate the effectiveness of the proposed ILC algorithms and initial conditions.

Notations. Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space, $\mathbb{R}^{n\times m}$ the set of all $n\times m$ real matrices, $\mathbb{Z}$ the set of all integers, $\mathbb{N}_+$ the positive integer, I_n the $n\times n$ identity matrix, $|\cdot|$ the usual Euclidean norm. The notation $*$ represents the elements below the main diagonal of a symmetric matrix, and $\oplus$ stands for the direct sum.

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2 FORMATTING INSTRUCTIONS

Consider the following linear time-invariant discrete system

$$\begin{cases} \boldsymbol{x}_k(t+1) &= A\boldsymbol{x}_k(t) + B\boldsymbol{u}_k(t), \\ \boldsymbol{y}_k(t) &= C\boldsymbol{x}_k(t), t \geq 0, \end{cases} \quad (1)$$

where $t \in [0, T]$ is the discrete-time axis, $k \geq 0$ is the iteration axis, $\boldsymbol{x}_k(t) \in \mathbb{R}^n$ is the state, $\boldsymbol{u}_k(t) \in \mathbb{R}^m$ is the input, and $\boldsymbol{y}_k(t) \in \mathbb{R}^p$ is the output. Furthermore, $A \in \mathbb{R}^{n\times n}, B \in \mathbf{R}^{n\times m}, C \in \mathbb{R}^{p\times n}$ are real matrices and $\boldsymbol{x}_k(0)$ is the initial states. The initial state, and the packet loss process satisfy the following assumptions:

A1) For simplicity, no data at $k = 0$ is assumed to be lost.

A2) For each t, the packet loss is random without obeying any certain probability distribution. However, there exists a integer $\kappa < \infty$ such that during successive K iterations, at least in one iteration its data is successfully sent back (see below Fig.1 and [16]). Let the function $k_i(j) : \mathbb{Z} \to \mathbb{Z}$ be the effective iterations during $0 \leq k \leq j$ and time $t = i$. Then, we have

$$\lim_{j\to\infty} k_i(j) = \infty. \quad (2)$$

A3) The initial states $\boldsymbol{x}_k(0)$ satisfying

$$\sum_{k=0}^{\infty} |\boldsymbol{x}_{k+1}(0) - \boldsymbol{x}_k(0)| < \infty. \quad (3)$$

In the absence of ambiguity, the function $k_i(j)$ is abbreviated as factor k_i.

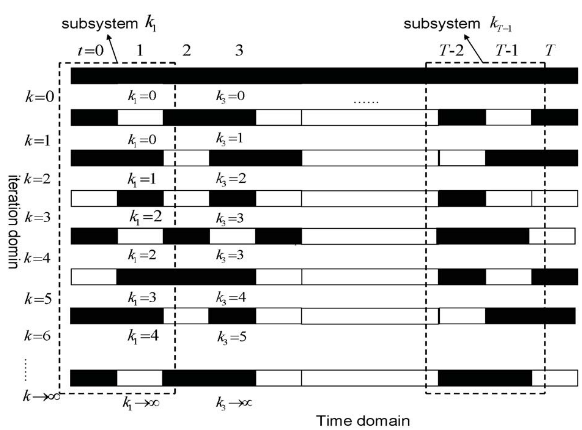

Figure 1: Diagram of ILC including packet loss

Remark 2.1 *The restriction A1) requires that the data is complete at $k = 0$, which is not necessary just facilitates the analysis. The assumption A2) indicates that for any time t, the effective iterations will approach infinite, i.e., $k_t \to \infty$, as the iteration axis $k \to \infty$, which further ensures that the perfect tracking can be achieved under the packet loss. From (3), we see that the initial state $\boldsymbol{x}_k(0)$ is convergent.*

Next, the linear system (1) is decomposed into $T - 1$ subsystems

$$\Sigma_{k_t} : \begin{cases} \boldsymbol{x}_{k_t}(t+1) &= A\boldsymbol{x}_{k_t}(t) + B\boldsymbol{u}_{k_t}(t), \\ \boldsymbol{y}_{k_t}(t) &= C\boldsymbol{x}_{k_t}(t). \end{cases} \quad (4)$$

It is worth noting that each subsystem Σ_{k_t} has only two time points t and $t + 1$ (see Fig.1). To handle iteration-varying initial state and packet loss satisfying assumption A1)-A3) ,the following ILC law is proposed for the subsystem Σ_{k_i}:

$$\begin{cases} \boldsymbol{u}_{k_t+1}(t) &= \boldsymbol{u}_{k_t}(t) + \Delta\boldsymbol{u}_{k_t}(t), \\ \boldsymbol{u}_{k_t=0}(t) &= \boldsymbol{u}^0(t), \end{cases} \quad (5)$$

where Δu is undetermined and denotes the modification of control input, $\boldsymbol{u}^0$ denotes the initial input.
Let $\boldsymbol{e}_{k_t}(t) = \boldsymbol{y}_r(t) - \boldsymbol{y}_{k_t}(t), \boldsymbol{\eta}_{k_t}(t) = \boldsymbol{x}_{k_t+1}(t-1) - \boldsymbol{x}_{k_t}(t-1)$, where $\boldsymbol{y}_r(t)$ denotes the reference. Then, the closed-loop subsystem (4) and (5) can be presented as

$$\begin{bmatrix} \boldsymbol{\eta}_{k_t}(t+1) \\ \boldsymbol{e}_{k_t+1}(t) \end{bmatrix} = \begin{bmatrix} A & 0 \\ -CA & I_p \end{bmatrix} \begin{bmatrix} \boldsymbol{\eta}_{k_t}(t) \\ \boldsymbol{e}_{k_t}(t) \end{bmatrix} + \begin{bmatrix} B \\ -CB \end{bmatrix} \Delta\boldsymbol{u}_{k_t}(t-1). \quad (6)$$

Clearly, the ILC system (6) is a 2-D Rosser model, and $\boldsymbol{e}_0(t), \boldsymbol{e}_0(t+1)$, and $\boldsymbol{\eta}_{k_t}(t), k_t \geq 0$ are the initial states.

Remark 2.2 *By using the properties of the solutions of 2-D Roesser model, necessary and sufficient conditions were presented in [8, 15] for the convergence of ILC rules based on the condition $i.i.c.$, i.e., $\boldsymbol{x}_k(0) = \boldsymbol{x}(0), \forall k \geq 0$. In [14], the robustness of ILC rules to a kind of variable initial states, i.e., $|\boldsymbol{x}_k(0) - \boldsymbol{x}(0)| \leq \delta, \forall k \geq 0$ was discussed, but the bound of robustness is normally too large.*

Remark 2.3 *It is very difficult (even impossible) to handle the initial condition problem in ILC by using the properties of the solutions of 2-D Roesser model. A natural thought is to use 2-D system stability theory. Many stability criteria of 2-D Roesser model (see [1, 3]) are established on a sufficient and necessary condition (see below Lemma 3.1) which is only valid for any diagonal initial states. It follows that these stability criteria are unavailable because the initial states of 2-D ILC model (6) are orthogonal. Therefore, It is necessary to deeply investigate the stability criteria of 2-D Roesser model under the orthogonal initial states.*

3 2-D system Theory

Consider the following 2-D Roesser model

$$\begin{bmatrix} \boldsymbol{x}^h(i+1,j) \\ \boldsymbol{x}^v(i,j+1) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}^h(i,j) \\ \boldsymbol{x}^v(i,j) \end{bmatrix}, \quad (7)$$

where $\boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}^h(i,j) \\ \boldsymbol{x}^v(i,j) \end{bmatrix} \in \mathbb{R}^n$ is the state, where $\boldsymbol{x}^h \in \mathbb{R}^{n_1}, \boldsymbol{x}^v \in \mathbb{R}^{n_2}, n_1 + n_2 = n$ represent the horizontal and vertical states, respectively; $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ is the system matrix with the submatrices $A_{ij}, i, j = 1, 2$ of appropriate dimensions. In general, there are two types of initial states:

Diagonal initial states:

$$\boldsymbol{x}(i,-i), \quad \forall i \in \mathbb{Z}. \tag{8}$$

Orthogonal initial states:

$$\boldsymbol{x}^v(i,0), \boldsymbol{x}^h(0,j), \quad \forall i,j \geq 0. \tag{9}$$

Let

$$X_r = \{\boldsymbol{x}(i,j) : i+j=r\}, |X_r| = \sup_{i\in\mathbb{Z}} |\boldsymbol{x}(r-i,i)|.$$

Definition 3.1 *(see Definition in [2]) 2-D Roesser model (7) is asymptotically stable if assuming $|X_0|$ finite, $|X_r| \to 0$ as $i \to \infty$.*

Lemma 3.1 *(see Theorem 1 in [2]) 2-D Roesser model (7) under diagonal initial states (8) is asymptotically stable if and only if*

$$\det \begin{bmatrix} I_{n_1} - z_1^{-1}A_{11} & -z_1^{-1}A_{12} \\ -z_2^{-1}A_{21} & I_{n_2} - z_2^{-1}A_{22} \end{bmatrix} \neq 0, \quad \forall |z_1^{-1}| \leq 1, |z_1^{-1}| \leq 1. \tag{10}$$

Note that the initial states of 2-D ILC model (6) are orthogonal. Then, the stability criteria established on the basis of (10) cannot be directly used to handle the initial condition problem of 2-D system (1) and (5). For the orthogonal initial states, we introduce the follow definition:

Definition 3.2 *(see Definition 4.2 in [9]) 2-D Roesser model (7) is asymptotically stable if $\lim_{i+j\to\infty} |\boldsymbol{x}(i,j)| = 0, \forall i,j \geq 1$ and initial states satisfy $\lim_{i\to\infty} |\boldsymbol{x}^v(i,0)| = 0, \lim_{j\to\infty} |\boldsymbol{x}^h(0,j)| = 0$.*

Lemma 3.2 *(see Theorem 1.1 in [9]) A solution to the Roesser model (7) with initial states $\boldsymbol{x}^h(0,j), \boldsymbol{x}^v(i,0), i,j \geq 0$ is given by*

$$\begin{aligned} \boldsymbol{x}(i,j) = & \sum_{k_1=0}^{i} \Phi^{k_1,j} \begin{bmatrix} 0 \\ \boldsymbol{x}^v(i-k_1,0) \end{bmatrix} \\ + & \sum_{k_2=0}^{j} \Phi^{i,k_2} \begin{bmatrix} \boldsymbol{x}^h(0,j-k_2) \\ 0 \end{bmatrix}, \end{aligned} \tag{11}$$

where $\Phi^{i,j}$ denotes the state transition matrix defined by

- $\Phi^{i,j} = \Phi^{1,0}\Phi^{i-1,j} + \Phi^{0,1}\Phi^{i,j-1}$
- $\Phi^{i,j} = 0, \forall i < 0$ or $j < 0$
- $\Phi^{0,0} = I_n, \Phi^{1,0} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix}$,

  $\Phi^{0,1} = \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix}$

Theorem 3.1 *2-D Roesser model (7) under orthogonal initial states (9) is asymptotically stable if the condition (10) holds and*

$$\sum_{i=0}^{\infty} |\boldsymbol{x}^v(i,0)| < \infty, \sum_{j=0}^{\infty} |\boldsymbol{x}^h(0,j)| < \infty. \tag{12}$$

Proof. Let

$$U(z_1^{-1}, z_2^{-1}) = \begin{bmatrix} I_{n_1} - z_1^{-1}A_{11} & -z_1^{-1}A_{12} \\ -z_2^{-1}A_{21} & I_{n_2} - z_2^{-1}A_{22} \end{bmatrix},$$

$$\mathcal{P}_a = \left\{(z_1^{-1}, z_2^{-1}) \in \mathbb{C} \times \mathbb{C} : |z_1^{-1}| \leq a, |z_2^{-1}| \leq a\right\}.$$

Since the algebraic curve of $det(U(z_1^{-1}, z_2^{-1})) = 0$ and the set $\mathcal{P}_1$ are close, then the condition (10) implies that there exists $\varepsilon > 0$ such that the rational matrix $U(z_1^{-1}, z_2^{-1})$ is inverted in the set $\mathcal{P}_{1+\varepsilon}$.
Note that (see Theorem 4.8 and Theorem A.2.4 in [9])

$$\begin{bmatrix} I_{n_1} - z_1^{-1}A_{11} & -z_1^{-1}A_{12} \\ -z_2^{-1}A_{21} & I_{n_2} - z_2^{-1}A_{22} \end{bmatrix}^{-1} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \Phi^{i,j} z_1^{-i} z_2^{-j}.$$

Then, we have that the series $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \Phi^{i,j} z_1^{-i} z_2^{-j}$ is absolutely convergent for $|z_1^{-1}| \leq 1+\varepsilon, |z_2^{-1}| \leq 1+\varepsilon$. It implies that there exist numbers $M \geq 0$ and $\lambda \in [0,1)$ such that

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \lambda^{-(i+j)} |\Phi^{i,j}| \leq M.$$

It follows that

$$|\Phi^{i,j}| \leq M\lambda^{(i+j)}. \tag{13}$$

Note that $\lim_{i\to\infty} \sum_{k_2=0}^{j} |\Phi^{i,k_2}||\boldsymbol{x}^h(0,j-k_2)| \leq \lim_{i\to\infty} M\lambda^i \sum_{k_2=0}^{j} |\boldsymbol{x}^h(0,j-k_2)|\lambda^{k_2} = 0$ and

$$\begin{aligned} & \sum_{i=0}^{\infty}\sum_{k_1=0}^{i} |\Phi^{k_1,j}||\boldsymbol{x}^v(i-k_1,0)| \\ \leq\ & M\lambda^j \sum_{i=0}^{\infty}\sum_{k_1=0}^{i} \lambda^{k_1}|\boldsymbol{x}^v(i-k_1,0)| \\ \leq\ & M\lambda^j \sum_{i=0}^{\infty}\lambda^i \sum_{k_1=0}^{\infty} |\boldsymbol{x}^v(k_1,0)| < \infty \end{aligned}$$

which implies $\lim_{i\to\infty} \sum_{k_1=0}^{i} |\Phi^{k_1,j}||\boldsymbol{x}^v(i-k_1,0)| = 0$. Then, we have

$$\lim_{i\to\infty} |\boldsymbol{x}(i,j)| = 0, \forall j \geq 0$$

With the same process, we also can get $\lim_{j\to\infty} |\boldsymbol{x}(i,j)| = 0, \forall i \geq 0$. Therefore, we have $|\boldsymbol{x}(i,j)| \to 0$ as $i+j \to \infty$.
□
Clearly, from Lemma 3.2, Theorem 3.1 can be rewritten as:

Corollary 3.1 *Suppose 2-D Roesser model (7) satisfying boundary condition (12). The 2-D system under orthogonal initial states (9) is asymptotically stable if and only if the condition (10) holds.*

The condition (10) only is available for any diagonal initial states rather than any orthogonal initial states, because $\Phi^{1,j} = 0, \forall j \geq 0$ is necessary to obtain $\lim_{j\to\infty} |\boldsymbol{x}(1,j)| = 0$ for any finite $x^h(0,j)$. However, all stability criterion, established based on the condition (10) and diagonal initial states, are available for the orthogonal initial states satisfying condition (12). Thus, from [1], we have the 2-D Lyapunov equation without proof:

Theorem 3.2 *Suppose 2-D Roesser model (7) satisfying boundary condition (12). Then, the 2-D system under orthogonal initial states (9) is asymptotically stable if there exist symmetric positive definite matrices* $W_1 \in \mathbb{R}^{n_1 \times n_1}, W_2 \in \mathbb{R}^{n_2 \times n_2}, W = W_1 \oplus W_2$ *such that*

$$A^T W A - W < 0. \quad (14)$$

In order to facilitate the ILC design based on above 2-D stability criteria, the following corollary is given.

Corollary 3.2 *The 2-D Roesser model (7) satisfies* $\sum_{j=0}^{\infty} |\boldsymbol{x}(i,j)| < \infty$ *for any* $i \in \mathbb{N}_+$ *if* $\sum_{j=0}^{\infty} |\boldsymbol{x}^h(0,j)| < \infty$ *and the condition (10) (or 2-D Lyapunov equation (14) holds.*

Proof. From the proof of Theorem 3.1, it is not hard to see that $\sum_{j=0}^{\infty} |\boldsymbol{x}(i,j)| < \infty$ for any $i \in \mathbb{N}_+$ if $\sum_{j=0}^{\infty} |\boldsymbol{x}^h(0,j)| < \infty$ and the condition (10) holds. Then, following the assertion of Theorem 3.2, the corollary is easily proved. □

Remark 3.1 *Proposition 1 in [7] and Theorem 4.8, Theorem A.2.4 in [9] play an important role in the proof of Theorem 3.1. The boundary condition (12) is similar to that in [20, 21] for 2-D FM modles. Furthermore, the LMI condition (14) could be used to design state-feedback and output-feedback ILC laws.*

4 2-D system based ILC algorithm

For the subsystem (4), it is natural to choose the ILC law (5) as

$$\begin{aligned} \boldsymbol{u}_{k_t+1}(t) = \ & \boldsymbol{u}_{k_t}(t) + K_1[\boldsymbol{x}_{k_t+1}(t) - \boldsymbol{x}_{k_t}(t)] \\ & + K_2[\boldsymbol{y}_r(t+1) - \boldsymbol{y}_{k_t}(t+1)]. \end{aligned} \quad (15)$$

It follows that

$$\boldsymbol{\Delta} u_{k_t}(t-1) = K_1 \boldsymbol{\eta}_{k_t}(t) + K_2 \boldsymbol{e}_{k_t}(t), \quad (16)$$

which is a state-feedback controller of the 2-D ILC system (6).

Now, let us determine the control input $\boldsymbol{u}_k(t)$ of the system (1) for the packet loss. In order to make the control input $\boldsymbol{u}_k(t)$ (1) equal to $\boldsymbol{u}_{k_t(k)}(t)$ in (15) for the subsystem (4), the control input $\boldsymbol{u}_k(t)$ of the system (1) must be operated as follows:

- If the data at t, k is not lost, then

$$\begin{aligned} \boldsymbol{u}_k(t) = \ & \boldsymbol{u}_{k-1}(t) + K_1[\boldsymbol{x}_k(t) - \boldsymbol{x}_{k-1}(t)] \\ & + K_2[\boldsymbol{y}_r(t+1) - \boldsymbol{y}_{k-1}(t+1)]. \end{aligned} \quad (17)$$

- If the data at t, k is lost, then

$$\begin{aligned} & \boldsymbol{u}_k(t) = \boldsymbol{u}_{k-1}(t), \boldsymbol{x}_k(t) = \boldsymbol{x}_{k-1}(t), \\ & \boldsymbol{y}_k(t+1) = \boldsymbol{y}_{k-1}(t+1). \end{aligned} \quad (18)$$

Therefore, the $T-1$ closed-loop subsystem (4) and (15) is equivalent to the linear system (1) implemented by the following 2-D system based ILC algorithm.

Finally, let us use the 2-D Lyapunov equation (14) to determine the controller gains K_1, K_2.

Theorem 4.1 *Suppose that linear system (1) satisfies the assumptions A1)-A3). The ILC algorithm in Table 1 can be used to achieve the perfect tracking of linear system (1), if there exists symmetric positive definite matrices* $P_1 \in \mathbb{R}^{n \times n}, P_2 \in \mathbb{R}^{p \times p}$ *and real matrices* $Q_1 \in \mathbb{R}^{m \times n}, Q_2 \in \mathbb{R}^{m \times p}$ *such that*

$$\begin{bmatrix} -P_1 & 0 & \Sigma_{13} & BQ_2 \\ * & -P_2 & \Sigma_{23} & \Sigma_{24} \\ * & * & -P_1 & 0 \\ * & * & * & -P_2 \end{bmatrix} < 0 \quad (19)$$

where $\Sigma_{13} = AP_1 + BQ_1, \Sigma_{23} = -CAP_1 - CBQ_1, \Sigma_{24} = I_p - CBQ_2$*. The gains* K_1, K_2 *can be chosen as* $K_1 = Q_1 P_1^{-1}, K_2 = Q_2 P_2^{-1}$.

Proof. The proof consists of three parts.

Part I: The inequalities $\sum_{k_t=0}^{\infty} |\boldsymbol{\eta}_{k_t}(t+1)| < \infty$ *and* $\sum_{k_t=0}^{\infty} |\boldsymbol{e}_{k_t}(t+1)| < \infty$ *hold if* $\sum_{k_t=0}^{\infty} |\boldsymbol{\eta}_{k_t}(t)| < \infty$.

By using Shur complement, it is not hard to show that the closed-loop 2-D ILC subsystem (6) and (16) satisfies 2-D Lyapunov equation (14) if the LMI in (19) holds. Thus, from Corollary 3.2, the inequalities $\sum_{k_i=0}^{\infty} |\boldsymbol{\eta}_{k_t}(t+1)| < \infty$ and $\sum_{k_i=0}^{\infty} |\boldsymbol{e}_{k_t}(t+1)| < \infty$ hold if $\sum_{k_t=0}^{\infty} |\boldsymbol{\eta}_{k_t}(t)| < \infty$ and LMI (19) hold.

Part II: The inequality $\sum_{k_t=0}^{\infty} |\boldsymbol{\eta}_{k_{t+1}}(t+1)| < \infty$ *holds if* $\sum_{k_i=0}^{\infty} |\boldsymbol{\eta}_{k_t}(t+1)| < \infty$.

Note that the fact $\boldsymbol{\eta}(t+1, k_{t+1}(k)) = \boldsymbol{\eta}_{k_t(k)}(t+1)$ though $k_{t+1}(k)$ may be not equal to $k_t(k)$. Then, Part II is proved which denotes that the absolutely convergence of $\boldsymbol{\eta}$ can be maintained from the iteration index k_t to k_{t+1}.

Part III: The tracking error satisfies $\sum_{k_t=0}^{\infty} |\boldsymbol{e}_{k_t}(t+1)| < \infty$ *for any* $t \in [0, T-1]$*, if the initial state* $\boldsymbol{x}_k(0)$ *satisfies (3). It implies* $\lim_{k \to \infty} \boldsymbol{e}_k(t) = 0$ *for any* $t \in [1, T]$ *because* $k_t \to \infty$ *as* $k \to \infty$.

From the initial condition (3), we have $\sum_{k_0=0}^{\infty} |\boldsymbol{\eta}_{k_0}(0)| < \infty$. Then, from *Part I*, we have that $\sum_{k_0=0}^{\infty} |\boldsymbol{\eta}_{k_0}(1)| < \infty$ and $\sum_{k_0=0}^{\infty} |\boldsymbol{e}_{k_0}(1)| < \infty$. It follows from *Part II* that $\sum_{k_0=0}^{\infty} |\boldsymbol{\eta}_{k_1}(1)| < \infty$. Continue this procedure, we can obtain $\sum_{k_t=0}^{\infty} |\boldsymbol{e}_{k_t}(t+1)| < \infty$ for any $t \in [0, T-1]$. Therefore, the proof is completed. □

```
Algorithm 1
Given x_k(0), u_0(t), x_0(t), y_0(t);
FOR k = 1 to ∞
  FOR t = 0 to T − 1
    IF the state or output data is not lost at current t, k THEN
      u_k(t) = u_{k−1}(t) + K_1[x_k(t) − x_{k−1}(t)]
               +K_2[y_r(t + 1) − y_{k−1}(t + 1)];
      x_k(t + 1) = Ax_k(t) + Bu_k(t);
      y_k(t + 1) = Cx_k(t + 1);
    ELSE
      u_k(t) = u_{k−1}(t); x_k(t) = x_{k−1}(t);
      y_k(t + 1) = y_{k−1}(t + 1);
    END IF
  END FOR
END FOR
```

Table 1: ILC algorithm with packet loss

5 Numerical examples

Consider the linear system (1) with

$$A = \begin{bmatrix} -0.24 & 0.1 \\ 0.04 & -0.35 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0.12 \end{bmatrix},$$
$$C = \begin{bmatrix} 0.5 & -0.01 \end{bmatrix}$$

By solving the LMI (19), we have that ILC algorithm 1 in Table 1 with gains $K_1 = [0.2337 \ -0.0670]$ and $K_2 = 0.3949$ can fulfill the perfect tracking for any initial states and packet lass satisfying A1)-A3).
Assume the reference trajectory $\boldsymbol{y}_r(t) = 1.5 \sin 0.1t, t \in [0, 100]$, the initial states $\boldsymbol{x}_k(0) = [(-0.5)^k \ (-0.5)^k]^T$, $k \geq 0$ and initial inputs $\boldsymbol{u}_0(t) = t, t \in [0, 99]$. The tracking performances with various packet lose rate $0 \leq p < 1$, which are displayed in Fig.2, are evaluated by the following total square error of tracking:

$$\mathcal{E}(k) = \sum_{t=0}^{100} \|\boldsymbol{y}_r(t) - \boldsymbol{y}_k(t)\|^2, k \geq 0$$

From Fig.2, we see that the perfect tracking of linear discrete systems with variable initial states satisfying (12) can be achieved by the ILC rule (5) with $\Delta \boldsymbol{u}(t, k) = K_1 \boldsymbol{\eta}(t + 1, k) + K_2 \boldsymbol{e}(t + 1, k)$.
If there is a packet loss, according to the Algorithm 1, we can obtain $\boldsymbol{u}(t, k) = \boldsymbol{u}(t, k-1); \boldsymbol{x}(t, k) = \boldsymbol{x}(t, k-1); \boldsymbol{y}(t+1, k) = \boldsymbol{y}(t + 1, k - 1)$; then we can get the total square error of tracking at different packet loss rate of iteration in Fig.3.

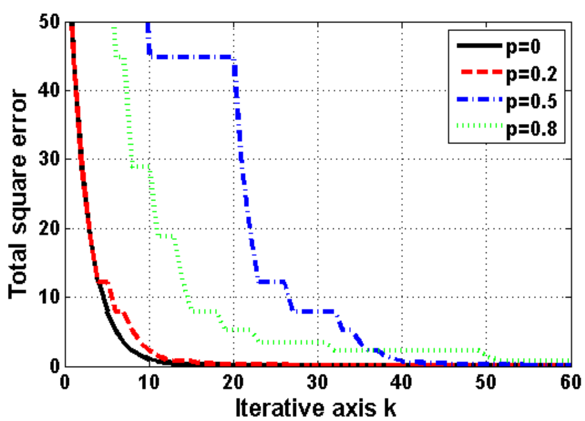

Figure 2: Total square errors of tracking of Examples at different packet loss rate along iteration axis

From Fig.2, we see that the perfect tracking can be achieved by the ILC algorithm 1 in Table 1 iteration-varying initial states and various packet loss rate. Especially, the tracking error at t, k does not change if the data at t, k is lost, which implies the good property of the ILC algorithm handling packet loss.

6 Conclusions

The main contribution of this study is to show that the perfect tracking can be achieved for the iteration-varying initial state and packet loss. In order to handle the iteration-varying initial state, the 2-D stability theory is studied under orthogonal initial states. It is shown that the 2-D Lyapunov equation can be used to deal with the absolutely convergent initial state (i.e., assumption A3)). To deal with the packet loss satisfying assumption A2), a novel approach, $\boldsymbol{u}_k(t) = \boldsymbol{u}_{k-1}(t), \boldsymbol{x}_k(t) = \boldsymbol{x}_{k-1}(t), \boldsymbol{y}_k(t + 1) = \boldsymbol{y}_{k-1}(t+1)$ is presented based on the transformed 2-D ILC system.

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