Cascade-Based Control of a Benchmark System

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Abstract: A cascade-based control method is proposed for the translational oscillator with a rotational actuator (TORA) system. Specifically, the original dynamic equations are transformed into a cascade form through some coordinate transformations. Then, a virtual control input is designed for the first subsystem and a deviation-based subsystem is introduced. Based on the deviation-based subsystem, a *linear filter* is presented and a nonlinear controller is proposed straightforwardly. Simulation results are provided to evaluate the performance of the proposed method in the presence of different initial conditions, uncertain parameters, and external disturbances.

Key Words: Underactuated mechatronics, benchmark system, TORA, cascade-based control

1 INTRODUCTION

In practice, underactuated mechanical systems, such as underactuated robots [1, 2], cranes [3–5], VTOL aircraft [6, 7], are widely utilized for their advantages including simple mechanical structure, low energy consumption, and high efficiency, and so on. The TORA system as a typical underactuated mechanical system is originally proposed as a simplified model of a dual-spin spacecraft to study the resonance capture phenomenon. Since it possesses many interesting challenges from view of global nonlinear control. Recently, it is used as a benchmark system for educational purposes or to investigate the performance of nonlinear control techniques.

During the last few decades, extensive research has been implemented on the benchmark TORA system. In particular, Bupp et al. describe the benchmark TORA system and give parameters for a nominal configuration [8]. Ref. [9] proposes an output regulation control method for the nonlinear benchmark system via a certainty-equivalence design, which only uses the information of the rotational proof mass position. In [10], on the basis of partial feedback linearization and integrator backstepping schemes, a nonlinear feedback control method is proposed for global stabilization of the oscillating eccentric rotor. However, the structure of the control law is significantly complex. In addition, many other control techniques, such as passivity-based control [11–14], output feedback control [15–18], saturation control [19], intelligent control [20], and other control strategies [21–24], are investigated.

The research presented here is based on using the cascade technology to address the stabilization problem of the TORA system. Compared with the integrator-backstepping-based control law, the structure of the presented method here is much simpler. Firstly, the original dynamic model of the TORA system is transformed into a cascade form consisting of two connected subsystems, and then a virtual control input is developed to stabilize the first subsystem and a deviation-based subsystem is constructed. Then, based on the deviation-based subsystem, a linear filter is designed to stabilize the deviation signal. Subsequently, an actual control method for the whole system is proposed straightforwardly. The corresponding stability of the subsystems and the whole system is guaranteed. Finally, simulation results are given to demonstrate the effectiveness and flexibility of the proposed control law in the presence of different initial conditions, uncertain system parameters, and extraneous disturbances.

The remaining parts of this paper are organized as follows. The dynamic model of the TORA system is given and some coordinate transformations are performed in Section 2. Controller design and stability analysis are carried out in Section 3. In Section 4, some simulation results are presented to examine the performance of the presented control law. Some conclusions are drawn in Section 5.

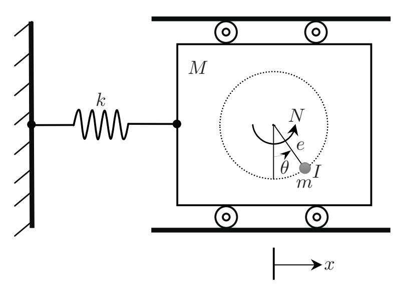

Figure 1: Structure of the TORA system.

2 DYNAMIC MODEL OF THE TORA SYSTEM

In this paper, we consider the benchmark TORA system as shown in Fig. 1, the equations of motion for the system are

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given by [10]:

$$(M+m)\ddot{x} + me(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) + kx = 0 \quad (1)$$

$$me\ddot{x}\cos\theta + (me^2 + I)\ddot{\theta} = N \quad (2)$$

where a cart with mass M is connected to a fixed wall by a linear spring of stiffness k; the proof mass actuator has mass m and moment of inertia I about its center of mass, and its center of mass is located a distance e from the axis about which it rotates; $x(t)$ and $\dot{x}(t)$ are the translational displacement and velocity of the cart, respectively; $\theta(t)$ and $\dot{\theta}(t)$ stand for the angular position and velocity of the rotational disk; N represents the control torque applied to the rotational disk. Our task is to design a control law that achieves disturbance rejection with respect to the translational displacement of the cart and stabilizes the system states to the equilibrium position asymptotically.
After suitable normalizations [8], the following normalized dimensionless equations can be obtained:

$$\dot{x}_1 = x_2 \quad (3a)$$

$$\dot{x}_2 = \frac{-x_1 + \epsilon x_4^2 \sin x_3}{1 - \epsilon^2\cos^2 x_3} - \frac{\epsilon\cos x_3}{1 - \epsilon^2\cos^2 x_3}u \quad (3b)$$

$$\dot{x}_3 = x_4 \quad (3c)$$

$$\dot{x}_4 = \frac{\epsilon\cos x_3(x_1 - \epsilon x_4^2\sin x_3)}{1 - \epsilon^2\cos^2 x_3} + \frac{1}{1 - \epsilon^2\cos^2 x_3}u \quad (3d)$$

where $0 < \epsilon < 1$ and x_1 represents the normalized position of the translational oscillator from the equilibrium point; $x_3 = \theta$ denotes the angle of the eccentric rotational proof mass; u is the normalized control torque applied to the proof mass. As in [10], we introduce the following variables:

$$\chi_1 = x_1 + \epsilon\sin x_3 \quad (4a)$$

$$\chi_2 = x_2 + \epsilon x_4\cos x_3 \quad (4b)$$

$$z_1 = x_3 \quad (4c)$$

$$z_2 = x_4 \quad (4d)$$

$$\upsilon = \frac{1}{1 - \epsilon^2\cos^2 z_1}\left\{\epsilon\cos z_1[\chi_1 - (1 + z_2^2)\epsilon\sin z_1] + u\right\} \quad (4e)$$

Then, we can rewrite equation (3) in a cascade form in terms of the new variables as

$$\dot{\chi}_1 = \chi_2 \quad (5a)$$

$$\dot{\chi}_2 = -\chi_1 + \epsilon\sin z_1 \quad (5b)$$

$$\dot{z}_1 = z_2 \quad (5c)$$

$$\dot{z}_2 = \upsilon \quad (5d)$$

where υ is the auxiliary control variable to be designed in the forthcoming section.

3 CONTROLLER DESIGN AND STABILITY ANALYSIS

In this section, our objective is to synthesize a nonlinear control law for (5). Toward this end, first, we view $z_1(t)$ as the control input of (5b) and seek for a desired control signal $z_{1d}(t)$ that stabilizes (5a) and (5b). To stabilize (5a) and (5b) with $z_{1d}(t)$ as the desired control variable, let the "virtual control input" as

$$z_{1d} = -\arctan(\alpha\chi_2) \quad (6)$$

where $\alpha \in \mathbb{R}^+$ is a positive constant, and choose a Lyapunov function as

$$V_\chi(t) = \frac{1}{2}\chi_1^2 + \frac{1}{2}\chi_2^2 \quad (7)$$

and taking the derivative of $V_\chi(t)$ with respect to time, and substituting the "virtual control input" $z_{1d}(t)$ in (6) into the resulting expression for $z_1(t)$, and then rearranging the resulting expression, one can obtain that

$$\dot{V}_\chi(t) = -\chi_2\epsilon\sin(\arctan(\alpha\chi_2)) \quad (8)$$

whichi is nonincreasing, and the closed-loop system (5a) and (5b) is stable in the sense of Lyapunov. It follows from LaSalle's invariance principle [25] that the states of the closed-loop system (5a) and (5b) will asymptotically converge to the original point. Additionally, it follows from aforementioned analysis that $z_{1d}(t)$ globally asymptotically stabilizes the closed-loop system (5a) and (5b).

Remark 1. *It is worthwhile to mention that, for the virtual control input $z_{1d}(t)$, one can conveniently replace the inverse tangent function with the hyperbolic tangent function $\tanh(*)$ or $(*)/\sqrt{1+(*)^2}$, with guaranteed stability and convergence properties.*

Considering $z_1(t)$ is not the real control variable, and the deviation from its desired value is defined as

$$\xi_1 := z_1 - z_{1d} = z_1 + \arctan(\alpha\chi_2) \quad (9)$$

and the corresponding time derivatives can be derived

$$\xi_2 := \dot{\xi}_1 = z_2 - \dot{z}_{1d} \quad (10)$$

$$\xi_3 := \dot{\xi}_2 = \dot{z}_2 - \ddot{z}_{1d} = \upsilon - \ddot{z}_{1d} \quad (11)$$

where the detailed expressions of $\dot{z}_{1d}(t)$ and $\ddot{z}_{1d}$ are

$$\dot{z}_{1d} = \alpha\frac{\chi_1 - \epsilon\sin z_1}{1 + (\alpha\chi_2)^2} \quad (12)$$

$$\ddot{z}_{1d} = \alpha\frac{\chi_2 - \epsilon z_2\cos z_1}{1 + (\alpha\chi_2)^2} + \alpha^3\frac{2\chi_2(\chi_1 - \epsilon\sin z_1)^2}{(1 + (\alpha\chi_2)^2)^2} \quad (13)$$

Then, the dynamic model (5) can be rewritten as the following cascade form:

$$\dot{\chi}_1 = \chi_2 \quad (14a)$$

$$\dot{\chi}_2 = -\chi_1 + \epsilon\sin(\xi_1 + z_{1d}) \quad (14b)$$

$$\dot{\xi}_1 = \xi_2 \quad (14c)$$

$$\dot{\xi}_2 = \upsilon - \ddot{z}_{1d} \quad (14d)$$

In order to stabilize the ξ-subsystem, i.e., (14c) and (14d), a linear filter is defined as follows:

$$\phi := \xi_2 + \beta\xi_1 \quad (15)$$

where $\beta \in \mathbb{R}^+$ is a positive constant, and the time derivative of which is

$$\dot{\phi} = \dot{\xi}_2 + \beta\dot{\xi}_1 = \upsilon - \ddot{z}_{1d} + \beta\xi_2 \quad (16)$$

On the basis of (16), a proper control signal is introduced for $\upsilon(t)$

$$\upsilon = -\kappa\phi + \ddot{z}_{1d} - \beta\xi_2 \quad (17)$$

where $\kappa \in \mathbb{R}^+$ is a positive control gain, and then one can derive the actual control input $u(t)$ in accordance with (4e) straightforwardly. To implement it to the original dynamic model of (3), by substituting (17) into (4e) for υ, one can obtain the detailed expression of the ultimate control input as follows:

$$\begin{aligned} u = & -\epsilon\cos z_1[\chi_1 - (1+z_2^2)\epsilon\sin z_1] \\ & + (1-\epsilon^2\cos^2 z_1)(-\kappa\phi + \ddot{z}_{1d} - \beta\xi_2) \end{aligned} \quad (18)$$

where $\xi_2, \ddot{z}_{2d}, \phi$ are defined in (10), (13) and (15), respectively.

For the presented nonlinear control method (18), we will analyze the stability of the origin of the closed-loop system (3) through the following theorem.

Theorem 1. *The proposed nonlinear control method provided in* (18) *ensures the origin of the closed-loop system is globally asymptotically stable in the sense that*

$$\lim_{t\to\infty}[x_1\ x_2\ x_3\ x_4]^{\mathrm{T}} = [0\ 0\ 0\ 0]^{\mathrm{T}} \quad (19)$$

Proof. To evaluate the stability of the origin of the closed-loop system (3), we could prove it through testifying the stability of the closed-loop system (14) equivalently. We view this system (14) as a cascade system which involves the χ-subsystem and the ξ-subsystem.

The global asymptotic stability of the origin of the χ-subsystem can be obtained from (7) and (8) via LaSalle's invariance principle [25].

In order to demonstrate the stability of the ξ-subsystem, by substituting the proposed auxiliary control method (17) into (16), one can obtain the following conclusion:

$$\dot{\phi} = -\kappa\phi \quad (20)$$

which implies that $\phi(t)$ tends to zero exponentially as t approaches infinity. Obviously, it follows from the linear filter (15) that $\lim_{t\to\infty}[\xi_1, \xi_2]^{\mathrm{T}} = [0, 0]^{\mathrm{T}}$. Hence, one can obtain that the origin of the ξ-subsystem is globally exponentially stable.

Since the right-hand of the χ-subsystem is globally Lipschitz and bounded, the origin of the χ-subsystem is globally asymptotically stable and the origin of the ξ-subsystem is globally exponentially stable. Hence, by invoking Theorem 6.2 of [26], we can conclude that the equilibrium point of the closed-loop cascade system (14) is globally asymptotically stable, that is,

$$\lim_{t\to\infty}[\chi_1\ \chi_2\ \xi_1\ \xi_2]^{\mathrm{T}} = [0\ 0\ 0\ 0]^{\mathrm{T}} \quad (21)$$

which further indicates

$$\lim_{t\to\infty}[x_1\ x_2\ x_3\ x_4]^{\mathrm{T}} = [0\ 0\ 0\ 0]^{\mathrm{T}} \quad (22)$$

where the equality (4) has been utilized. This ends the proof. □

4 NUMERICAL SIMULATION

In this section, a series of simulation tests will be performed to examine the control performance of the proposed method. For the following simulations, we consider the dimensionless dynamic model (3) and the control parameters of the controller are set to be $\alpha = 1.6, \beta = 0.6, \kappa = 0.6$. The simulation tests are divided into three groups, the details of each group will be given in the following subsections.

4.1 Stabilization of Different Initial Conditions

To evaluate the effectiveness of the presented method for different initial conditions, three sets (all with $\epsilon = 0.2$) of initial conditions are chosen.

Condition 1: $[x_1(0)\ x_2(0)\ x_3(0)\ x_4(0)]^{\mathrm{T}} = [1\ 0\ 0\ 0]^{\mathrm{T}}$;

Condition 2: $[x_1(0)\ x_2(0)\ x_3(0)\ x_4(0)]^{\mathrm{T}} = [0.5\ 0\ -0.5\ 0]^{\mathrm{T}}$;

Condition 3: $[x_1(0)\ x_2(0)\ x_3(0)\ x_4(0)]^{\mathrm{T}} = [-0.5\ 0\ 0.5\ 0]^{\mathrm{T}}$.

The results of this group are shown in Fig. 2. It can be observed from Fig. 2 that the system states are all stabilized to the origin points by the proposed control approach even the initial conditions are different, which approves the superior performance of the proposed method.

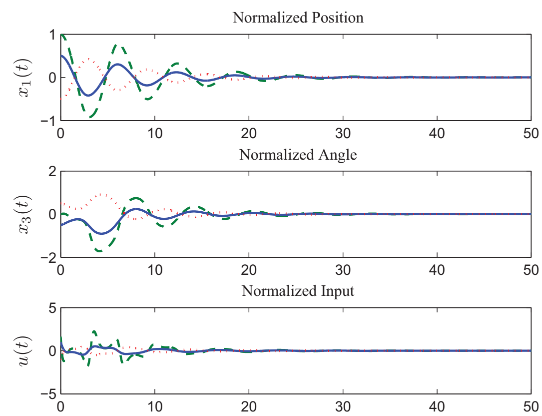

Figure 2: Simulation results (*dashed line:* Condition 1; *solid line:* Condition 2; *dotted line:* Condition 3).

4.2 Robustness to Uncertain Parameters

In order to demonstrate the robustness of the proposed method to uncertain system parameters, the parameter ϵ is chosen as $\epsilon = 0.1, \epsilon = 0.2, \epsilon = 0.3$ with the initial conditions $[x_1(0)\ x_2(0)\ x_3(0)\ x_4(0)]^{\mathrm{T}} = [1\ 0\ 0\ 0]^{\mathrm{T}}$, respectively. The results of this group are exhibited in Fig. 3. By comparing these results, it is clear that uncertain system parameters hardly influence the overall control responses of the closed-loop system.

4.3 Robustness to External Disturbances

To illustrate the robustness of the designed controller in the presence of external disturbances. We add different external disturbances to the cart during the control process.

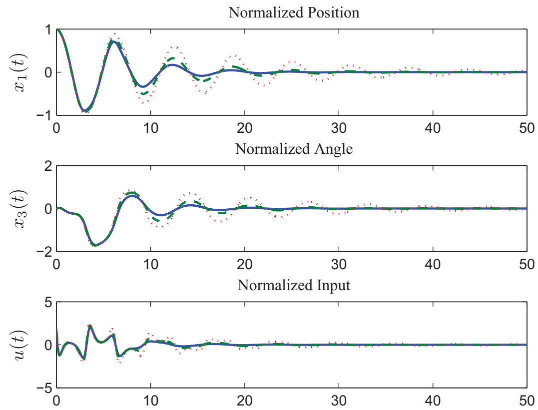

Figure 3: Simulation results (*dotted line:* $\epsilon = 0.1$; *dashed line:* $\epsilon = 0.2$; *solid line:* $\epsilon = 0.3$).

Specifically, impulse disturbance from 1 to 1.01 and random disturbance from 25 to 26 (both with an amplitude of 1) are added to the translational oscillation cart and the initial conditions are set to be zero. The corresponding simulation results are recorded in Fig. 4, from which one can find that the added external disturbances are suppressed and eliminated quickly under the control of the presented method.

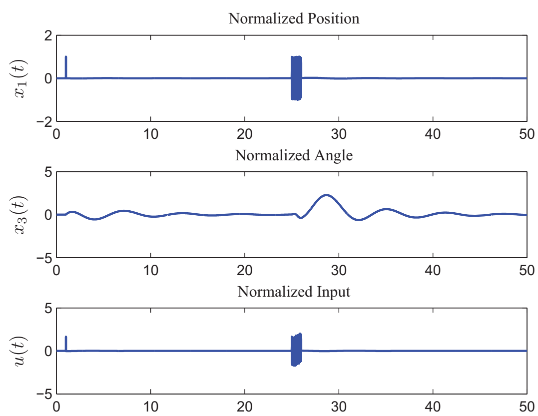

Figure 4: Simulation results in the presence of disturbances.

5 CONCLUSIONS

A cascade-based control method has been proposed for the stabilization problem of the benchmark TORA system. Specifically, the original dynamic equations are transformed into a cascade form through some coordinate transformations. Then, the cascade system is divided into two subsystems, and a virtual control input is proposed for the first subsystem, and a deviation-based subsystem is constructed. Subsequently, a nonlinear controller is proposed for the whole system and the stability of each subsystems and the whole system are guaranteed via rigorous mathematical analysis. Simulation results are provided to evaluate the performance of the proposed method and its robustness against uncertain parameters and external disturbances.

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