# Global State-feedback Control for A Class of High-order Stochastic Upper-triangular Systems with Input Time-varying Delay

Liang Liu[1,2], Yifan Zhang[3]

1. College of Engineering, Bohai University, Jinzhou, 121013, China

2. School of Automation, Nanjing University of Science and Technology, Nanjing, 210094, China
E-mail: smithll@163.com

3. College of Mathematics and Physics, Bohai University, Jinzhou, 121013, China
E-mail: trustyifan@163.com

**Abstract:** This paper deals with the global state-feedback control for a class of high-order stochastic upper-triangular systems with input time-varying delay. By extending the homogeneous domination approach to stochastic case, skillfully choosing an appropriate Lyapunov-Krasoviskii functional and the low gain scale, the delay-independent state-feedback controller is constructed to guarantee that the closed-loop system is globally asymptotically stable in probability. The simulation example verifies the effectiveness of the proposed design scheme.

**Key Words:** High-order stochastic upper-triangular systems, Time-varying delay, Homogeneous domination approach

## 1 INTRODUCTION

This paper will consider the following high-order stochastic upper-triangular systems with input time-varying delay

$$\begin{aligned} dx_i &= x_{i+1}^p dt + f_i(\tilde{x}_{i+2}, u(t-\tau(t)))dt \\ &\quad + g_i^T(\tilde{x}_{i+1}, u(t-\tau(t)))d\omega, \\ dx_n &= u^p dt,\ i = 1, \cdots, n-1, \end{aligned} \tag{1}$$

where $x = (x_1, \cdots, x_n)^T \in R^n$ and $u \in R$ are the system state and control input, respectively. $\tilde{x}_i = (x_i, \cdots, x_n)$, $i = 2, \cdots, n$, $\tau(t) : R^+ \to [0, \tau_0]$ is time-varying delay. The system power $p \in R_{odd}^{>2} =: \{q \in R^+ : q > 2$ is a ratio of odd integers$\}$. $\omega$ is an $m$-dimensional standard Wiener process defined on the complete probability space $(\Omega, \mathcal{F}, P)$ with $\Omega$ being a sample space, $\mathcal{F}$ being a filtration and $P$ being a probability measure. The mappings $f_i$ and $g_i$ are assumed to be locally Lipschitz with $f_i(0, 0) = 0$ and $g_i(0, 0) = 0$, $i = 1, \cdots, n-1$.

When the diffusion term $g_i(\cdot, \cdot) = 0$, system (1) reduces to the deterministic upper-triangular (i.e. feedforward) nonlinear systems with input delay. In the past few years, the study on the deterministic upper-triangular systems with input delay has been achieved lots of interesting results. [1] studied the globally uniformly asymptotically and locally exponentially stabilization for a class of nonlinear feedforward systems with input delay. [2] proposed the global asymptotic state-feedback control scheme for feedforward nonlinear systems with input delay. [3] considered the state feedback controller design for a class of high-order upper-triangular nonlinear systems with input delay. On the basis of the reduction method to remove the input delay, [4] proposed the state feedback controller using dynamic gain for input-delayed systems. By adopting a linear observer and dynamic sampled-data output feedback control law, [5] solved output-feedback control for upper-triangular systems with delay in the input. [6] investigated state-feedback control for feedforward input-delay nonlinear systems with ratios of odd integer powers.

It should be pointed out that the aforementioned results are not consider the effect of stochastic noise. For the case of the diffusion term $g_i(\cdot, \cdot) \neq 0$, [7] firstly investigated the state-feedback control for stochastic upper-triangular systems with state delay and the system power $p = 1$. [8, 9] further considered the asymptotic stabilization of stochastic feedforward systems with input delay and the system power $p = 1$. [10] dealt with the state-feedback control for high-order stochastic upper-triangular systems without delay. However, for the high-order stochastic upper-triangular systems with input time-varying delay (1), there are no relevant results so far. The main purpose of this paper is to solve the global state-feedback control problem for high-order stochastic upper-triangular systems (1). To solve this problem, we will combine the homogeneous domination approach with stochastic nonlinear time-delay system stability criterion. By skillfully choosing an appropriate Lyapunov-Krasovskii functional and the low gain scale, a delay-independent state-feedback controller is explicitly designed to render the closed-loop system is globally asymptotically stable in probability.

The paper is organized as follows. Section 2 gives some preliminary results. The design of state-feedback controller is proposed in Section 3. Section 4 states the main result of this paper, following a simulation example in Section 5. Section 6 concludes this paper.

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## 2 MATHEMATICAL PRELIMINARIES

The following notations, definitions and lemma are to be used in this paper. $\mathcal{C}([-d,0];R^n)$ denotes the space of continuous $R^n$-value functions on $[-d,0]$ endowed with the norm $\|\cdot\|$ defined by $\|f\| = \sup_{x\in[-d,0]}|f(x)|$ for $f \in \mathcal{C}([-d,0];R^n)$; $\mathcal{C}^b_{\mathcal{F}_0}([-d,0];R^n)$ denotes the family of all $\mathcal{F}_0$-measurable bounded $\mathcal{C}([-d,0];R^n)$-valued random variables $\xi = \{\xi(\theta) : -d \le \theta \le 0\}$.

Consider the stochastic time-delay system

$$\begin{aligned} dx(t) &= f(x(t), x(t-d(t)), t)dt \\ &\quad + g(x(t), x(t-d(t)), t)d\omega, \quad \forall t \ge 0, \end{aligned} \tag{2}$$

with initial data $\{x(\theta) : -d \le \theta \le 0\} = \xi \in \mathcal{C}^b_{\mathcal{F}_0}([-d,0];R^n)$, where $d(t) : R_+ \to [0,d]$ is a Borel measurable function, $\omega$ is an $m$-dimensional standard Wiener process defined on the complete probability space $(\Omega, \mathcal{F}, P)$, $f$ and $g$ are locally Lipschitz with $f(0,0,t) = 0$ and $g(0,0,t) = 0$.

**Definition 1**[11]: For any given $V(x(t),t) \in \mathcal{C}^{2,1}$ associated with system (2), the differential operator $\mathcal{L}$ is defined as $\mathcal{L}V = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}f + \frac{1}{2}\mathrm{Tr}\left\{g^T\frac{\partial^2 V}{\partial x^2}g\right\}$.

**Lemma 1**[11]: For system (2), if there exist a function $V(x(t),t) \in \mathcal{C}^{2,1}(R^n \times [-d,\infty); R_+)$, two class $\mathcal{K}_\infty$ functions $\alpha_1$, $\alpha_2$ and a class $\mathcal{K}$ function $\alpha_3$ such that

$$\begin{aligned} &\alpha_1(|x(t)|) \le V(x(t),t) \le \alpha_2\left(\sup_{-d\le s\le 0}|x(t+s)|\right), \\ &\mathcal{L}V(x(t),t) \le -\alpha_3(|x(t)|), \end{aligned}$$

then there exists a unique solution on $[-d,\infty)$ for (2), the equilibrium $x(t) = 0$ is globally asymptotically stable in probability and $P\{\lim_{t\to\infty}|x(t)| = 0\} = 1$.

## 3 STATE-FEEDBACK CONTROLLER DESIGN

We need the following assumptions for system (1).

**Assumption 1:** For $i = 1,\cdots,n-1$, there exist positive constants $A_1$ and $A_2$ such that for any $p \in R^{\ge 2}_{odd}$ and $\sigma \in [d_1, +\infty)$,

$$\begin{aligned} |f_i| &\le A_1\left(\sum_{j=i+2}^{n}|x_j|^{\frac{r_i+\sigma}{r_j}} + |u(t-\tau(t))|^{\frac{r_i+\sigma}{r_{n+1}}}\right), \\ |g_i| &\le A_2\left(\sum_{j=i+1}^{n}|x_j|^{\frac{2r_i+\sigma}{2r_j}} + |u(t-\tau(t))|^{\frac{2r_i+\sigma}{2r_{n+1}}}\right), \end{aligned}$$

where $d_1 = \frac{(2p^{n-1}-1)(p-1)}{p^n-1}, r_1 = 1, r_{i+1} = \frac{r_i+\sigma}{p}, i = 1,\cdots,n$.

**Assumption 2:** The time-varying delay $\tau(t)$ satisfies $\dot{\tau}(t) \le \delta < 1$ for a constant $\delta$.

Without loss of generality, we assume $\sigma = \frac{n_1}{m_1}$ with $n_1$ being an even integer and $m_1$ being an odd integer in this paper. Under this assumption, one can conclude that $r_i$ is an odd number.

**Remark 1:** Clearly, system (1) satisfying Assumption 1 is a class of high-order stochastic upper-triangular systems with input time-varying delay. As discussed in [2], [3], [5]-[10], Assumption 1 is a general and frequently-used condition. However, system (1) studied in this paper is more general than [10]. Furthermore, as $u = 0$ in $f_i$ and $x_{i+1} = u = 0$ in $g_i$, Assumption 1 changes into the same assumption used in [10].

Introducing the following coordinate transformation

$$z_1 = x_1,\ z_i = \frac{x_i}{N^{\kappa_i}},\ v = \frac{u}{N^{\kappa_{n+1}}},\ i = 1,\cdots,n, \tag{3}$$

system (1) can be rewritten as

$$\begin{aligned} dz_i &= \left(Nz^p_{i+1} + \frac{f_i}{N^{\kappa_i}}\right)dt + \frac{g_i^T}{N^{\kappa_i}}d\omega, \\ dz_n &= Nv^p dt,\ i = 1,\cdots,n-1, \end{aligned} \tag{4}$$

where $\kappa_1 = 0$, $\kappa_j = \frac{\kappa_{j-1}+1}{p}$, $j = 2,\cdots,n+1$, $0 < N < 1$ is a gain to be determined later.

Before giving the design of state-feedback controller for system (4), we give a key lemma, whose role is to guarantee Lemma 3 and Step 1 in the proof of Theorem 1.

**Lemma 2:** For any $p \in R^{\ge 2}_{odd}$ and $\sigma \in [d_1, +\infty)$, there exist $l \ge 1$ and $\mu \in R^+_{odd} =: \{\bar{q} \in R_+ : \bar{q}$ is a ratio of odd integers$\}$ such that $r_n + \sigma \ge \mu \ge \max_{1\le i\le n}\{2r_i, \frac{r_i+\sigma}{l}\}$.

*Proof*: The detailed proof can be referred Lemma 5 in [11].

Step 1: Choosing $\xi_1 = z_1^{\frac{\mu}{r_1}}$ and $V_1(z_1) = \int_{z_1^*}^{z_1}(s^{\frac{\mu}{r_1}} - z_1^{*\frac{\mu}{r_1}})^{\frac{q_1}{\mu}}ds$, where $z_1^* = 0$, $q_1 = 4l\mu - \sigma - r_1$. By Definition 1 and (4), one has $\mathcal{L}V_1 = Nz_1^{\frac{q_1}{r_1}}z_2^p + F_1 + G_1$, where $F_1 = \frac{\partial V_1}{\partial z_1}f_1$, $G_1 = \frac{1}{2}\mathrm{Tr}\{g_1\frac{\partial^2 V_1}{\partial z_1^2}g_1^T\}$. The first virtual controller

$$z_2^* = -\beta_1^{\frac{r_2}{\mu}}\xi_1^{\frac{r_2}{\mu}},\ \beta_1 = c_{11}^{\frac{\mu}{r_2 p}},\ c_{11} > 0, \tag{5}$$

leads to $\mathcal{L}V_1 \le -Nc_{11}\xi_1^{4l} + N\xi_1^{\frac{q_1}{\mu}}(z_2^p - z_2^{*p}) + F_1 + G_1$.

Step $i (i = 2,\cdots,n)$: In this step, one can obtain the similar property for the $i$th subsystem, which presents in the following lemma.

**Lemma 3:** Suppose that at step $i-1$, there are a $\mathcal{C}^2$, positive definite and proper Lyapunov function $V_{i-1}(\bar{z}_{i-1})$ and a series of virtual controllers $z_1^*,\cdots,z_i^*$ defined by

$$\begin{aligned} &z_1^* = 0,\ \xi_1 = z_1^{\frac{\mu}{r_1}}, \\ &z_k^* = -\beta_{k-1}^{\frac{r_k}{\mu}}\xi_{k-1}^{\frac{r_k}{\mu}},\ \xi_k = z_k^{\frac{\mu}{r_k}} - z_k^{*\frac{\mu}{r_k}},\ k = 2,\cdots,i, \end{aligned} \tag{6}$$

such that

$$\begin{aligned} \mathcal{L}V_{i-1}(\bar{z}_{i-1}) &\le -N\sum_{j=1}^{i-1}c_{i-1,j}\xi_j^{4l} + N\xi_{i-1}^{\frac{q_{i-1}}{\mu}}(z_i^p - z_i^{*p}) \\ &\quad + F_{i-1} + G_{i-1}, \end{aligned} \tag{7}$$

where $q_{i-1} = 4l\mu - \sigma - r_{i-1}$, $\beta_j$, $c_{i-1,j}$, $j = 1,\cdots,i-1$, are positive constants, $F_{i-1} = \sum_{j=1}^{i-1}\frac{\partial V_{i-1}}{\partial z_j}\frac{f_j}{N^{\kappa_j}}$, $G_{i-1} = \sum_{p,q=1}^{i-1}\frac{1}{2}\mathrm{Tr}\{\frac{g_p}{N^{\kappa_p}}\frac{\partial^2 V_{i-1}}{\partial z_p \partial z_q}\frac{g_q^T}{N^{\kappa_q}}\}$. Then the $i$th Lyapunov function

$$\begin{aligned} V_i(\bar{z}_i) &= V_{i-1}(\bar{z}_{i-1}) + U_i(\bar{z}_i), \\ U_i(\bar{z}_i) &= \int_{z_i^*}^{z_i}(s^{\frac{\mu}{r_i}} - z_i^{*\frac{\mu}{r_i}})^{\frac{q_i}{\mu}}ds, \\ q_i &= 4l\mu - \sigma - r_i, \end{aligned} \tag{8}$$

is $\mathcal{C}^2$, positive definite and proper, and there is a virtual control law $z_{i+1}^* = -\beta_i^{\frac{r_{i+1}}{\mu}} \xi_i^{\frac{r_{i+1}}{\mu}}$ such that

$$\mathcal{L}V_i(\bar{z}_i) \le -N\sum_{j=1}^{i} c_{ij}\xi_j^{4l} + N\xi_i^{\frac{q_i}{\mu}}(z_{i+1}^p - z_{i+1}^{*p}) + F_i + G_i, \tag{9}$$

where $G_i = \sum_{p,q=1}^{i} \frac{1}{2}\mathrm{Tr}\{\frac{g_p}{N^{\kappa_p}} \frac{\partial^2 V_i}{\partial z_p \partial z_q} \frac{g_q^T}{N^{\kappa_q}}\}$, $F_i = \sum_{j=1}^{i} \frac{\partial V_i}{\partial z_j} \frac{f_j}{N^{\kappa_j}}$.

*Proof*: Firstly, we prove that $V_i(\bar{z}_i)$ is $\mathcal{C}^2$. By (6) and (8), it is easy to deduce that

$$\frac{\partial U_i}{\partial z_i} = \xi_i^{\frac{q_i}{\mu}}, \quad \frac{\partial^2 U_i}{\partial z_i^2} = \frac{q_i}{r_i} z_i^{\frac{\mu}{r_i}-1} \xi_i^{\frac{q_i}{\mu}-1},$$

$$\frac{\partial^2 U_i}{\partial z_i \partial z_j} = \frac{\partial^2 U_i}{\partial z_j \partial z_i} = -\frac{q_i}{\mu} \frac{\partial z_i^{*\frac{\mu}{r_i}}}{\partial z_j} \xi_i^{\frac{q_i}{\mu}-1},$$

$$\frac{\partial U_i}{\partial z_j} = -\frac{q_i}{\mu} \frac{\partial z_i^{*\frac{\mu}{r_i}}}{\partial z_j} \int_{z_i^*}^{z_i} (s^{\frac{\mu}{r_i}} - z_i^{*\frac{\mu}{r_i}})^{\frac{q_i}{\mu}-1} ds,$$

$$\frac{\partial^2 U_i}{\partial z_j^2} = -\frac{q_i}{\mu} \frac{\partial^2 z_i^{*\frac{\mu}{r_i}}}{\partial z_j^2} \int_{z_i^*}^{z_i} (s^{\frac{\mu}{r_i}} - z_i^{*\frac{\mu}{r_i}})^{\frac{q_i}{\mu}-1} ds + \frac{q_i(q_i-\mu)}{\mu^2} \left(\frac{\partial z_i^{*\frac{\mu}{r_i}}}{\partial z_j}\right)^2 \int_{z_i^*}^{z_i} (s^{\frac{\mu}{r_i}} - z_i^{*\frac{\mu}{r_i}})^{\frac{q_i}{\mu}-2} ds,$$

$$\frac{\partial^2 U_i}{\partial z_j \partial z_k} = \frac{\partial^2 U_i}{\partial z_k \partial z_j} = \frac{q_i(q_i-\mu)}{\mu^2} \frac{\partial z_i^{*\frac{\mu}{r_i}}}{\partial z_j} \frac{\partial z_i^{*\frac{\mu}{r_i}}}{\partial z_k} \cdot \int_{z_i^*}^{z_i} (s^{\frac{\mu}{r_i}} - z_i^{*\frac{\mu}{r_i}})^{\frac{q_i}{\mu}-2} ds, \ j,k = 1,\cdots,i-1, \tag{10}$$

Using Lemma 2, $\frac{\partial z_i^{*\frac{\mu}{r_i}}}{\partial z_j} = -\beta_{i-1}\cdots\beta_j \frac{\mu}{r_j} z_j^{\frac{\mu-r_j}{r_j}}$ and $\frac{\partial^2 z_i^{*\frac{\mu}{r_i}}}{\partial z_j^2} = -\beta_{i-1}\cdots\beta_j \frac{\mu(\mu-r_j)}{r_j^2} z_j^{\frac{\mu-2r_j}{r_j}}$, one gets $\frac{q_i}{\mu} - 2 \ge 0$, $\frac{\mu-r_j}{r_j} \ge 1$ and $\frac{\mu}{r_j} - 2 \ge 0$, from which and (10), one concludes that $U_i(\bar{z}_i)$ is $\mathcal{C}^2$, and then $V_i(\bar{z}_i)$ is also $\mathcal{C}^2$.

Next, we prove that $V_i(\bar{z}_i)$ is positive definite and proper.

Case I: When $z_i^* \le z_i$, using (8), $\frac{\mu}{r_i} \in R_{odd}^{\ge 2}$ and Lemma 2 in [11], one gets

$$U_i(\bar{z}_i) \ge 2^{\frac{q_i}{\mu}-\frac{q_i}{r_i}} \int_{z_i^*}^{z_i} (s - z_i^*)^{\frac{q_i}{r_i}} ds = \frac{2^{\frac{q_i}{\mu}-\frac{q_i}{r_i}} r_i}{4l\mu - \sigma} (z_i - z_i^*)^{\frac{4l\mu-\sigma}{r_i}}. \tag{11}$$

Case II: When $z_i^* \ge z_i$, (11) can be obtained similarly. Hence, $V_i(\bar{z}_i) \ge V_{i-1}(\bar{z}_{i-1}) + \bar{\delta}_i (z_i - z_i^*)^{\frac{4l\mu-\sigma}{r_i}}$, which implies that $V_i(\bar{z}_i)$ is positive definite and proper, where $\bar{\delta}_i$ is a positive constant.

At last, we prove inequality (9). From Definition 1, (6)-(8) and (10), it follows that

$$\mathcal{L}V_i(\bar{z}_i) \le -N\sum_{j=1}^{i-1} c_{i-1,j}\xi_j^{4l} + N\xi_i^{\frac{q_i}{\mu}} z_{i+1}^p + F_i + G_i + N\xi_{i-1}^{\frac{q_{i-1}}{\mu}}(z_i^p - z_i^{*p}) - \frac{q_i}{\mu} N \sum_{j=1}^{i-1} \frac{\partial z_i^{*\frac{\mu}{r_i}}}{\partial z_j} z_{j+1}^p \cdot \int_{z_i^*}^{z_i} (s^{\frac{\mu}{r_i}} - z_i^{*\frac{\mu}{r_i}})^{\frac{q_i}{\mu}-1} ds. \tag{12}$$

In the following, we deal with the last two terms on the right-hand side of (12).

By (6) and Lemmas 2,4 in [11], one has

$$z_i^p - z_i^{*p} \le \begin{cases} 2^{1-\frac{r_i p}{\mu}} |\xi_i|^{\frac{r_i p}{\mu}}, \text{ when } \frac{r_i p}{\mu} \le 1, \\ \bar{\gamma}_{i1} |\xi_{i-1}|^{\frac{r_i p}{\mu}} + \tilde{\gamma}_{i1} |\xi_i|^{\frac{r_i p}{\mu}}, \text{ when } \frac{r_i p}{\mu} \ge 1, \end{cases} \tag{13}$$

from which and Lemma 3 in [11], one leads to

$$\begin{aligned} &\xi_{i-1}^{\frac{q_{i-1}}{\mu}} (z_i^p - z_i^{*p}) \\ &\le |\xi_{i-1}|^{\frac{q_{i-1}}{\mu}} \left( \bar{\gamma}_{i1} |\xi_{i-1}|^{\frac{r_i p}{\mu}} + \gamma_{i1} |\xi_i|^{\frac{r_i p}{\mu}} \right) \\ &\le l_{i,i-1,1} \xi_{i-1}^{4l} + \alpha_{i1} \xi_i^{4l}, \end{aligned} \tag{14}$$

where $\gamma_{i1} = \max\{2^{1-\frac{r_i p}{\mu}}, \tilde{\gamma}_{i1}\}$, $\bar{\gamma}_{i1}$, $\tilde{\gamma}_{i1}$, $l_{i,i-1,1}$ and $\alpha_{i1}$ are positive constants.

Using (6) and Lemmas 2-3 in [11], one obtains

$$\begin{aligned} &-\frac{q_i}{\mu} \sum_{j=1}^{i-1} \frac{\partial z_i^{*\frac{\mu}{r_i}}}{\partial z_j} z_{j+1}^p \int_{z_i^*}^{z_i} (s^{\frac{\mu}{r_i}} - z_i^{*\frac{\mu}{r_i}})^{\frac{q_i}{\mu}-1} ds \\ &\le \tilde{c} \sum_{j=1}^{i-1} |z_j|^{\frac{\mu}{r_j}-1} |z_{j+1}|^p |\xi_i|^{\frac{4l\mu-\sigma}{\mu}-1} \\ &\le \sum_{j=1}^{i-1} l_{ij2} \xi_j^{4l} + \alpha_{i2} \xi_i^{4l}, \end{aligned} \tag{15}$$

where $\tilde{c}$, $l_{ij2}$ and $\alpha_{i2}$ are positive constants.

Taking

$$c_{ij} = \begin{cases} c_{i-1,j} - l_{ij2} > 0, \ j = 1,\cdots,i-2, \\ c_{i-1,i-1} - l_{i,i-1,1} - l_{i,i-1,2} > 0, \ j = i-1, \\ c_{ii} > 0, \ j = i, \end{cases}$$

$$z_{i+1}^* = -\beta_i^{\frac{r_{i+1}}{\mu}} \xi_i^{\frac{r_{i+1}}{\mu}}, \ \beta_i = (c_{ii} + \alpha_{i1} + \alpha_{i2})^{\frac{\mu}{r_{i+1} p}},$$

and substituting (14)-(15) into (12), inequality (9) holds.

At step $n$, choosing $V_n(\bar{z}_n) = V_{n-1}(\bar{z}_{n-1}) + \int_{z_n^*}^{z_n} (s^{\frac{\mu}{r_n}} - z_n^{*\frac{\mu}{r_n}})^{\frac{q_n}{\mu}} ds$ and

$$\begin{aligned} v &= z_{n+1}^* = -\beta_n^{\frac{r_{n+1}}{\mu}} \xi_n^{\frac{r_{n+1}}{\mu}} \\ &= -(\bar{\beta}_n z_n^{\frac{\mu}{r_n}} + \cdots + \bar{\beta}_1 z_1^{\frac{\mu}{r_1}})^{\frac{r_{n+1}}{\mu}}, \end{aligned} \tag{16}$$

from (9) and (16), it follows that

$$\begin{aligned} &\mathcal{L}V_n(\bar{z}_n) \\ &\le -N\sum_{i=1}^{n} c_{ni}\xi_i^{4l} + N\xi_n^{\frac{q_n}{\mu}}(v^p - z_{n+1}^{*p}) + F_n + G_n \\ &= -N\sum_{i=1}^{n} c_{ni}\xi_i^{4l} + \frac{\partial V_n}{\partial z} F + \frac{1}{2}\mathrm{Tr}\left\{ G \frac{\partial^2 V_n}{\partial z^2} G^T \right\}, \end{aligned} \tag{17}$$

where $\xi_n = z_n^{\frac{\mu}{r_n}} - z_n^{*\frac{\mu}{r_n}}$, $\bar{\beta}_i = \beta_n \cdots \beta_i$, $c_{ni}$, $i = 1,\cdots,n$, are positive constants, $F_n = \sum_{j=1}^{n-1} \frac{\partial V_n}{\partial z_j} \frac{f_j}{N^{\kappa_j}}$, $G_n = \sum_{p,q=1}^{n-1} \frac{1}{2}\mathrm{Tr}\{\frac{g_p}{N^{\kappa_p}} \frac{\partial^2 V_n}{\partial z_p \partial z_q} \frac{g_q^T}{N^{\kappa_q}}\}$, $F = (f_1, \cdots, \frac{f_{n-1}}{N^{\kappa_{n-1}}}, 0)^T$

$G = (g_1, \cdots, \frac{g_{n-1}}{N^{\kappa_{n-1}}}, 0)$. The system (4) and (16) can be expressed as the following compact form

$$dz = NE(z)dt + F(z, v(t-\tau(t)))dt + G^T(z, v(t-\tau(t)))d\omega, \quad (18)$$

where $z = (z_1, \cdots, z_n)^T$, $E(z) = (z_2^p, \cdots, z_n^p, v^p)^T$, $F$ and $G$ are defined as in (17). Introducing the dilation weight $\triangle = (\underbrace{r_1, r_2, \cdots, r_n}_{\text{for } z_1, \cdots, z_n})$, from Definition 2 in [7], we know that $V_n(z)$ is homogeneous of degree $4l\mu - \sigma$.

**Remark 2:** In this paper, by reasonably selecting parameters $l$ and $\mu$ in Lemma 2, Lemma 3 effectively avoids the zero-division problem of $\frac{\partial^2 z_i^{*\frac{\mu}{r_i}}}{\partial z_j^2}$. Please see eq.(10) below for the details.

## 4 STABILITY ANALYSIS

We state the main result of this paper as follows.

**Theorem 1**: If Assumptions 1-2 hold for high-order stochastic upper-triangular systems with input time-varying delay (1), under the state-feedback controller $u = N^{\kappa_{n+1}} v$ and (16), then (i) The closed-loop system has a unique solution on $[-\tau_0, +\infty)$; (ii) The equilibrium at the origin of the closed-loop system is globally asymptotically stable in probability.

*Proof*: The proof of Theorem 1 can be divided into the following two steps.

Step 1: We first prove that $v^p$ in (16) is $\mathcal{C}^1$. From (16), it follows that

$$\frac{\partial v^p}{\partial z_i} = -\frac{r_{n+1}p}{r_i}\bar{\beta}_i z_i^{\frac{\mu - r_i}{r_i}}(\bar{\beta}_n z_n^{\frac{\mu}{r_n}} + \cdots + \bar{\beta}_1 z_1^{\frac{\mu}{r_1}})^{\frac{r_n+\sigma-\mu}{\mu}}, \quad i = 1, \cdots, n. \quad (19)$$

By Lemma 2, one has $\frac{\mu - r_i}{r_i} \geq 1$, $\frac{\mu}{r_i} \geq 2$, $\frac{r_n+\sigma-\mu}{\mu} \geq 0$, from which and (19), we know that $\frac{\partial v^p}{\partial z_i}$ is continuous, then $v^p$ is $\mathcal{C}^1$. Since $f_i$, $g_i$, $(i = 1, \ldots, n-1)$ are assumed to be locally Lipschitz, the system consisting of (4) and (16) satisfies the locally Lipschitz condition.

Step 2: By Lemma 3 in [7] and (17), there exists a positive constant $c_{01}$ such that

$$\frac{\partial V_n}{\partial z} NE(z) \leq -c_{01} N \|z\|_\triangle^{4l\mu}, \quad (20)$$

where $\|\cdot\|_\triangle$ is a homogeneous 2-norm as defined in [7]. By Assumption 1, (3) and $0 < N < 1$, we have

$$\begin{aligned}
&\frac{f_i}{N^{\kappa_i}} \\
&\leq A_1 N^{1+\rho_{i1}} \left( \sum_{j=i+2}^{n} |z_j|^{\frac{r_i+\sigma}{r_j}} + |v(t-\tau(t))|^{\frac{r_i+\sigma}{r_{n+1}}} \right) \\
&\leq \bar{A}_1 N^{1+\rho_{i1}} \left( \|z\|_\triangle^{r_i+\sigma} + \|z(t-\tau(t))\|_\triangle^{r_i+\sigma} \right), \quad (21)
\end{aligned}$$

where $0 < \rho_{i1} = \frac{p(p-1)}{(p-1)+\sigma(p^{i+1}-1)} \leq 1$ and $\bar{A}_1$ are positive constants. With the help of Lemmas 2-4 in [7] and (21), one can lead to

$$\begin{aligned}
&\frac{\partial V_n}{\partial z} F(z, v(t-\tau(t))) \\
&\leq \sum_{i=1}^{n-1} \left| \frac{\partial V_n}{\partial z_i} \right| \left| \frac{f_i}{N^{\kappa_i}} \right| \\
&\leq \tilde{c}_{02} N^{1+\rho_{01}} \sum_{i=1}^{n-1} \|z\|_\triangle^{q_i} (\|z\|_\triangle^{r_i+\sigma} + \|z(t-\tau(t))\|_\triangle^{r_i+\sigma}) \\
&\leq N^{1+\rho_{01}} (c_{02}\|z\|_\triangle^{4l\mu} + \bar{c}_{02}\|z(t-\tau(t))\|_\triangle^{4l\mu}), \quad (22)
\end{aligned}$$

where $c_{02}$, $\tilde{c}_{02}$, $\bar{c}_{02}$ and $\rho_{01} = \min_{1 \leq i \leq n-1}\{\rho_{i1}\}$ are positive constants. According to Assumption 1 and (3), $0 < N < 1$, one has

$$\begin{aligned}
&\frac{g_i}{N^{\kappa_i}} \\
&\leq A_2 N^{\frac{1}{2}+\rho_{i2}} \left( \sum_{j=i+1}^{n} |z_j|^{\frac{2r_i+\sigma}{2r_j}} + |v(t-\tau(t))|^{\frac{2r_i+\sigma}{2r_{n+1}}} \right) \\
&\leq \bar{A}_2 N^{\frac{1}{2}+\rho_{i2}} (\|z\|_\triangle^{r_i+\frac{\sigma}{2}} + \|z(t-\tau(t))\|_\triangle^{r_i+\frac{\sigma}{2}}), \quad (23)
\end{aligned}$$

where $0 < \rho_{i2} = \frac{p-1}{2((p-1)+\sigma(p^i-1))} \leq \frac{1}{2}$ and $\bar{A}_2$ are positive constants. From Lemmas 2-4 in [7] and (23), one gets

$$\begin{aligned}
&\frac{1}{2}\mathrm{Tr}\left\{ G(z, v(t-\tau(t))) \frac{\partial^2 V_n}{\partial z^2} G^T(z, v(t-\tau(t))) \right\} \\
&\leq \frac{1}{2} m\sqrt{m} \sum_{i,j=1}^{n-1} \left| \frac{\partial^2 V_n}{\partial z_i \partial z_j} \right| \left| \frac{g_i}{N^{\kappa_i}} \right| \left| \frac{g_j}{N^{\kappa_j}} \right| \\
&\leq \tilde{c}_{03} N^{1+\rho_{02}} \sum_{i,j=1}^{n-1} \|z\|_\triangle^{q_i - r_j} (\|z(t-\tau(t))\|_\triangle^{r_i+\frac{\sigma}{2}} \\
&\quad + \|z\|_\triangle^{r_i+\frac{\sigma}{2}})(\|z\|_\triangle^{r_j+\frac{\sigma}{2}} + \|z(t-\tau(t))\|_\triangle^{r_j+\frac{\sigma}{2}}) \\
&\leq N^{1+\rho_{02}} \left( c_{03}\|z\|_\triangle^{4l\mu} + \bar{c}_{03}\|z(t-\tau(t))\|_\triangle^{4l\mu} \right), \quad (24)
\end{aligned}$$

where $\rho_{02} = \min_{1 \leq i,j \leq n-1}\{\rho_{i2}+\rho_{j2}\}$, $c_{03}$, $\bar{c}_{03}$ and $\tilde{c}_{03}$ are positive constants. Consider the entire Lyapunov function

$$V(z) = V_n(z) + \frac{(\bar{c}_{02}+\bar{c}_{03})}{1-\delta} N^{1+\rho_0} \int_{t-\tau(t)}^{t} \|z(s)\|_\triangle^{4l\mu} ds, \quad (25)$$

where $0 < \rho_0 = \min\{\rho_{01}, \rho_{02}\} \leq 1$. By Definition 1, $0 < N < 1$, (18), (20), (22) and (24), one deduces that

$$\begin{aligned}
\mathcal{L}V(z) &\leq -c_{01}N\|z\|_\triangle^{4l\mu} + \left( c_{02} + c_{03} + \frac{\bar{c}_{02}+\bar{c}_{03}}{1-\delta} \right) \\
&\quad \cdot N^{1+\rho_0}\|z\|_\triangle^{4l\mu} \\
&= -\left( c_{01} - \left( c_{02} + c_{03} + \frac{\bar{c}_{02}+\bar{c}_{03}}{1-\delta} \right) N^{\rho_0} \right) \\
&\quad \cdot N\|z\|_\triangle^{4l\mu}. \quad (26)
\end{aligned}$$

By selecting $0 < N < \min\{1, (\frac{c_{01}}{c_{02}+c_{03}+\frac{\bar{c}_{02}+\bar{c}_{03}}{1-\delta}})^{\frac{1}{\rho_0}}\}$, (26) becomes $\mathcal{L}V(z) \leq -c_0\|z\|_\triangle^{4l\mu}$, where $c_0$ is a positive constant. With the help of Lemma 1, one can conclude that the system consisting of (4) and (16) has a unique solution on $[-\tau_0, \infty)$, $z(t) = 0$ is globally asymptotically stable in probability and $P\{\lim_{t\to\infty} |z(t)| = 0\} = 1$. Since (3) is an equivalent transformation, Theorem 1 holds.

**Remark 3:** (i) In the stability analysis, by using homogeneous domination approach, the effect of input time-varying delay was skillfully dealt by inequalities (21) and (23). (ii) Due to the appearance of input delay, how to construct an appropriate Lyapunov- Krasovskii functional $V(z)$ to satisfy (26) is not an easy work.

## 5 A SIMULATION EXAMPLE

Consider the following stochastic nonlinear system

$$dx_1 = x_2^{\frac{7}{3}}dt + \frac{1}{20}u^{\frac{49}{23}}(t-\tau(t))dt + \frac{1}{10}x_2 \sin x_2 d\omega,$$
$$dx_2 = u^{\frac{7}{3}}dt, \tag{27}$$

where $\tau(t) = \frac{1}{6}(1+\sin t)$. By $p = \frac{7}{3}$, we have $d_1 = \frac{11}{10}$, then $\sigma \in [\frac{11}{10}, +\infty)$. Taking $\sigma = 2$, then $r_1 = 1$, $r_2 = \frac{9}{7}$. It's easy to verify that Assumptions 1 and 2 are satisfied with $A_1 = \frac{1}{20}$, $A_2 = \frac{1}{10}$ and $\dot{\tau}(t) = \frac{1}{6}\cos t < 1$. In the following, we select $l = \frac{9}{7}$, $\mu = 3$, then Lemma 2 holds. Introducing the change of coordinates $z_1 = x_1$, $z_2 = \frac{x_2}{N^{\frac{3}{7}}}$, $v = \frac{u}{N^{\frac{30}{49}}}$, following the design procedure in Section 3, one obtains the state-feedback controller

$$u = -9.7936(N^{\frac{7}{23}}x_2^{\frac{7}{3}} + 2N^{\frac{30}{23}}x_1^3)^{\frac{23}{49}}. \tag{28}$$

In simulation, we choose $N = 0.1$, the initial values $x_1(0) = -4$, $x_2(0) = 3$. Figure 1 demonstrates the effectiveness of the proposed design scheme.

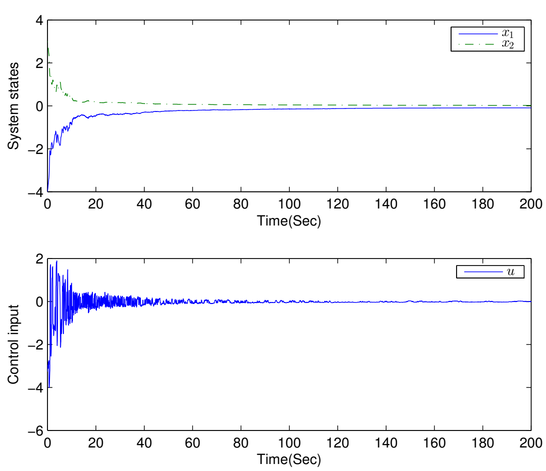


Figure 1: Responses of the closed-loop system (27)-(28).

## 6 CONCLUSION

This paper investigates the global state-feedback stabilization problem for a class of high-order stochastic upper-triangular systems with input time-varying delay. The designed state-feedback controller can guarantee that the closed-loop system is globally asymptotically stable in probability.

There exist some issues that need to be further considered: One is for Assumption 1, as $1 \le p < 2$, how to determine the range of $\sigma$ and design a stabilizing controller? Another is to solve the data-driven or fault diagnosis problem of system (1) discussed in [12, 13].

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